A Network Game with Attacker and Protector Entities *

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Abstract

Consider an information network with harmful procedures called *attackers* (e.g., viruses); each attacker uses a probability distribution to choose a node of the network to damage. Opponent to the attackers is the *system protector* scanning and cleaning from attackers some part of the network (e.g., an edge or a path), which it chooses independently using another probability distribution. Each attacker wishes to maximize the probability of escaping its cleaning by the system protector; towards a conflicting objective, the system protector aims at maximizing the expected number of cleaned attackers.

We model this network scenario as a non-cooperative strategic game on graphs. We focus on the special case where the protector chooses a single edge. We are interested in the associated *Nash equilibria,* where no network entity can unilaterally improve its local objective. We obtain the following results:

- No instance of the game possesses a pure Nash equilibrium.
- Every mixed Nash equilibrium enjoys a graph-theoretic structure, which enables a (typically exponential) algorithm to compute it.
- We coin a natural subclass of mixed Nash equilibria, which we call *matching Nash equilibria,* for this game on graphs. Matching Nash equilibria are defined using structural parameters of graphs, such as independent sets and matchings.
  - We derive a characterization of graphs possessing matching Nash equilibria. The characterization enables a linear time algorithm to compute a matching Nash equilibrium on any such graph with a given independent set and vertex cover.
  - Bipartite graphs are shown to satisfy the characterization. So, using a polynomial-time algorithm to compute a perfect matching in a bipartite graph, we obtain, as our main result, an efficient graph-theoretic algorithm to compute a matching Nash equilibrium on any instance of the game with a bipartite graph.

Throughout the paper, some missing proofs can be found in the attached Appendix. It may be read at the discretion of the Program Committee.

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1 Introduction

Motivation and Framework. Consider an information network represented by an undirected graph. The nodes of the network are insecure and vulnerable to infection. A *system protector* (e.g., antivirus software) is available in the system; however, its capabilities are limited. The system protector can guarantee safety only to a small part of the network, such as a path or even a single edge, which it may choose using a probability distribution. A collection of *attackers* (e.g., viruses or Trojan horses) are also present in the network. Each attacker chooses (via a separate probability distribution) a node of the network; the node is harmed unless it is covered by the system protector. Apparently, the attackers and the system protector have conflicting objectives. The system protector seeks to protect the network as much as possible, while the attackers wish to avoid being caught by the network protector so that they be able to damage the network. Thus, the system protector seeks to maximize the expected number of attackers it catches, while each attacker seeks to maximize the probability it escapes from the system protector.

Naturally, we model this scenario as a strategic game with two kinds of players: the *vertex players* representing the attackers, and the *edge player* representing the system protector. The Individual Cost of each player is the quantity to be maximized by the corresponding entity. We are interested in the *Nash equilibria* [6, 7] associated with this game, where no player can unilaterally improve its Individual Cost by switching to a more advantageous probability distribution. We focus on the simplest case where the edge player chooses a singe edge.

Summary of Results. Our results are summarized as follows:

- We prove that no instance of the game has a pure Nash equilibrium (pure NE) (Theorem 3.1).
- We then proceed to study mixed Nash equilibria (mixed NE). We provide a graph-theoretic characterization of mixed NE (Theorem 3.2). Roughly speaking, the characterization yields that the support of the edge player and the vertex players are an edge cover and a vertex cover of the graph and an induced subgraph of the graph, respectively. Given the supports, the characterization provides a system of equalities and inequalities to be satisfied by the probabilities of the players. Unfortunately, this characterization only implies an exponential time algorithm for the general case.
- We introduce *matching* Nash equilibria, which are a natural subclass of mixed Nash equilibria with a graph-theoretic definition (Definition 4.1). Roughly speaking, the supports of vertex players in a matching Nash equilibrium form together an independent set of the graph, while each vertex in the supports of the vertex players is incident to only one edge from the support of the edge player.
- We provide a characterization of graphs admitting a *matching* Nash equilibrium (Theorem 4.3). We prove that a *matching* Nash equilibrium can be computed in linear time for any graph satisfying the characterization once a *suitable* independent set is given for the graph.
- We finally consider bipartite graphs for which we show that they satisfy the characterization of *matching* Nash equilibria; hence, they always have one (Theorem 5.3). More importantly, we prove that a *matching* Nash equilibrium can be computed in polynomial time for bipartite graphs (Theorem 5.4).

Significance. Our work joins the booming area of *Algorithmic Game Theory*. Our work is the *first*[1] to model realistic scenarios about infected networks as a strategic game and study its associ-

[1]To the best of our knowledge, [1] is a single exception. It considers inoculation strategies for victims of viruses and establishes connections with variants of the Graph Partition problem.

ated Nash equilibria. Our results contribute towards answering the general question of Papadimitriou [10] about the complexity of Nash equilibria for our special game. Our results highlight a fruitful interaction between *Game Theory* and *Graph Theory*. We believe that our *matching* Nash equilibria (and extensions of them) will find further applications in other network games and establish themselves as a candidate Nash equilibrium for polynomial time computation in other settings as well.

2 Framework

Throughout, we consider an undirected graph $G(V, E)$, with $|V(G)| = \nu$ and $|E(G)| = \mu$. Given a set of vertices $X \subseteq V$, the graph $G\backslash X$ is obtained by removing from G all vertices of X and their incident edges. A graph H, is an *induced* subgraph of G, if $V(H) \subseteq V(G)$ and $(u, v) \in E(H)$, whenever $(u, v) \in E(G)$. Denote $\Delta(G)$ the maximum degree of the graph G. For any vertex $v \in V(G)$, denote $Neigh(v) = \{u : (u, v) \in E(G)\}$, the set of neighboring vertices of v. For a set of vertices $X \subseteq V$, denote $Neigh(X) = \{u \notin X : (u, v) \in E(G)$ for some $v \in X\}$. For all above properties of a graph G, when no confusion raises, we omit G.

2.1 The Model

An information network is represented as an undirected graph $G(V, E)$. The vertices represent the network hosts and the edges represent the communication links. We define a non-cooperative game $\Pi(G) = \langle \mathcal{N}, \{S_i\}_{i\in\mathcal{N}}, \{\mathsf{IC}\}_{i\in\mathcal{N}}\rangle$ as follows:

- The set of players is $\mathcal{N} = \mathcal{N}_{VP} \cup \mathcal{N}_{EP}$, where $\mathcal{N}_{VP}$ is a finite set of *vertex* players vp_i, $i \geq 1$, and $\mathcal{N}_{EP}$ is a singleton set of an *edge* player ep.
- The strategy set S_i of each player vp_i, $i \in \mathcal{N}_{VP}$, is V; the strategy set S_{ep} of the player ep is E. Thus, the strategy set $\mathcal{S}$ of the game is $\left(\underset{i \in N_{VP}}{\times} S_i\right) \times S_{ep} = V^{|\mathcal{N}_{ep}|} \times E$.
- Take any *strategy profile* $\vec{\mathbf{s}} = \langle s_1, \ldots, s_{|\mathcal{N}_{VP}|}, S_{ep}\rangle \in \mathcal{S}$, also called a *configuration*.
  - The *Individual Cost* of vertex player vp_i is a function $\mathsf{IC}_i : \mathcal{S} \to \{0, 1\}$ such that $\mathsf{IC}_i(\vec{\mathbf{s}}) = \begin{cases} 0, & s_i \in S_{ep} \\ 1, & s_i \notin S_{ep} \end{cases}$; intuitively, vp_i receives 1 if it is not caught by the edge player, and 0 otherwise.
  - The *Individual Cost* of the edge player ep is a function $\mathsf{IC}_{ep} : \mathcal{S} \to \mathbb{N}$ such that $\mathsf{IC}_{ep}(\vec{\mathbf{s}}) = |\{s_i : s_i \in S_{ep}\}|$.

The configuration $\vec{\mathbf{s}}$ is a *pure Nash equilibrium* [6, 7] (abbreviated as *pure NE*) if for each player $i \in \mathcal{N}$, it minimizes IC_i over all configurations $\vec{\mathbf{t}}$ that differ from $\vec{\mathbf{s}}$ only with respect to the strategy of player i.

A *mixed strategy* for player $i \in \mathcal{N}$ is a probability distribution over its strategy set S_i; thus, a mixed strategy for a vertex player (resp., edge player) is a probability distribution over vertices (resp., over edges) of G. A *mixed strategy profile* $\vec{\mathbf{s}}$ is a collection of mixed strategies, one for each player. Denote $P_{\vec{\mathbf{s}}}(ep, e)$ the probability that edge player ep chooses edge $e \in E(G)$ in $\vec{\mathbf{s}}$; denote $P_{\vec{\mathbf{s}}}(vp_i, v)$ the probability that vertex player vp_i chooses vertex $v \in V$ in $\vec{\mathbf{s}}$. Note $\sum_{v\in V} P_{\vec{\mathbf{s}}}(vp_i, v) = 1$ for each vertex player vp_i; similarly, $\sum_{e\in E} P_{\vec{\mathbf{s}}}(ep, e) = 1$. Denote $P_{\vec{\mathbf{s}}}(VP, v) = \sum_{i\in\mathcal{N}_{VP}} P_{\vec{\mathbf{s}}}(vp_i, v)$ the probability that vertex v is chosen by some vertex player in $\vec{\mathbf{s}}$.

The *support* of a player $i \in \mathcal{N}$ in the configuration $\vec{\mathbf{s}}$, denoted $D_{\vec{\mathbf{s}}}(i)$, is the set of pure strategies in its strategy set to which i assigns strictly positive probability in $\vec{\mathbf{s}}$. Denote $D_{\vec{\mathbf{s}}}(VP) = \bigcup_{i\in\mathcal{N}_{VP}} D_{\vec{\mathbf{s}}}(i)$; so, $D_{\vec{\mathbf{s}}}(VP)$ contains all pure strategies (that is, vertices) to which some vertex player assigns strictly positive probability. Let also $ENeigh_{\vec{s}}(v) = \{(u,v)E : (u,v) \in D_{\vec{s}}(ep)\}$; that is $ENeigh_{\vec{s}}(v)$ contains all edges incident to v that are included in the support of the edge player in $\vec{s}$.

A mixed strategic profile $\vec{s}$ induces an *Expected Individual Cost* IC_i for each player $i \in \mathcal{N}$, which is the expectation, according to $\vec{s}$, of its corresponding Individual Cost (defined previously for pure strategy profiles). The mixed strategy profile $\vec{s}$ is a *mixed Nash equilibrium* [6, 7] (abbreviated as mixed NE) if for each player $i \in \mathcal{N}$, it minimizes IC_i over all configurations $\vec{t}$ that differ from $\vec{s}$ only with respect to the mixed strategy of player i.

For the rest of this section, fix a mixed strategy profile $\vec{s}$. For each vertex $v \in V$, denote $Hit(v)$ the event that the edge player hits vertex v. So, the probability (according to $\vec{s}$) of $Hit(v)$ is $P_{\vec{s}}(Hit(v)) = \sum_{e \in ENeigh(v)} P_{\vec{s}}(ep, e)$. Define the minimum hitting probability $P_{\vec{s}}$ as $\min_v P_{\vec{s}}(Hit(v))$. For each vertex $v \in V$, denote $m_{\vec{s}}(v)$ the expected number of vertex players choosing v (according to $\vec{s}$). For each edge $e = (u,v) \in E$, denote $m_{\vec{s}}(e)$ the expected number of vertex players choosing either u or v; so, $m_{\vec{s}}(e) = m_{\vec{s}}(u) + m_{\vec{s}}(v)$. It is easy to see that for each vertex $v \in V$, $m_{\vec{s}}(v) = \sum_{i \in \mathcal{N}_{VP}} P_{\vec{s}}(vp_i, v)$. Define the maximum expected number of vertex players choosing e in $\vec{s}$ as $\max_e m_{\vec{s}}(e)$.

We proceed to calculate the Expected Individual Cost. Clearly, for the vertex player $vp_i \in \mathcal{N}_{VP}$,

$$\mathsf{IC}_i(\vec{s}) = \sum_{v \in V(G)} P_{\vec{s}}(vp_i, v) \cdot (1 - P_{\vec{s}}(Hit(v)) = \sum_{v \in V(G)} \left(P_{\vec{s}}(vp_i, v) \cdot (1 - \sum_{e \in ENeigh(v)} P_{\vec{s}}(ep, e) \right) \quad (1)$$

For the edge player ep,

$$\mathsf{IC}_{ep}(\vec{s}) = \sum_{e=(u,v) \in E(G)} P_{\vec{s}}(ep, e) \cdot (m_{\vec{s}}(u) + m_{\vec{s}}(v)) = \sum_{e=(u,v) \in E(G)} \left(P_{\vec{s}}(ep, e) \cdot (\sum_{i \in N_{VP}} P_{\vec{s}}(vp_i, u) + P_{\vec{s}}(v_i, v)) \right) \quad (2)$$

2.2 Background from Graph Theory

Throughout this section, we consider the (undirected) graph $G = G(V, E)$.
$G(V, E)$ is *bipartite* if its vertex set V can be partitioned as $V = V_1 \cup V_2$ such that each edge $e = (u, v) \in E$ has one of its vertices in V_1 and the other in V_2. Such a graph is often referred to as a V_1, V_2-bigraph. Fix a set of vertices $S \subseteq V$. The graph G is an *S-expander* if for every set $X \subseteq S$, $|X| \leq |Neigh_G(X)|$.

A set $M \subseteq E$ is a *matching* of G if no two edges in M share a vertex. Given a matching M, say that set $S \subseteq V$ is *matched in* M if for every vertex $v \in S$, there is an edge $(v, u) \in\in M$. The size of a maximum matching of G is denoted by $\alpha'(G)$. A *vertex cover* of G is a set $V' \subseteq V$ such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$. The size of a minimum vertex cover of G is denoted as $\beta(G)$. An *edge cover* of G is a set $E' \subseteq E$ such that for every vertex $v \in V$, there is an edge $(v, u) \in E'$. Say that an edge $(u, v) \in E$ (resp., a vertex $v \in V$) is *covered* by the vertex cover V' (resp., the edge cover E') if either $u \in V'$ or $v \in V'$ (resp., if there is an edge $(u, v) \in E'$). Otherwise, the edge (resp., the vertex) is not *covered* by the vertex cover (resp., the edge cover). A set $IS \subseteq V$ is an *independent set* of G if for all vertices $u, v \in IS$, $(u, v) \notin E$. The size of a maximum independent set of G is denoted as $\alpha(G)$. Clearly, $IS \subseteq V$ is an independent set of G if and only if the set $VC = V \backslash IS$ is a vertex cover of G.

We will use the following consequence of Hall's Theorem [4] (see also [2], Chapter 6) on the marriage problem.

Proposition 2.1 (Hall's Theorem) *A graph G has a matching M in which the vertex set $X \subseteq V$ is matched if and only if for each for each subset $S \subseteq X$, $|N(S)| \geq |S|$.*

3 Nash Equilibria

For pure Nash Equilibria, we prove:

Theorem 3.1 *If G contains more than one edges, then $\Pi(G)$ has no pure Nash Equilibrium.*

Proof. Consider any graph G with at least two edges and any configuration $\vec{\mathbf{s}}$ of $\Pi(G)$. Let e the edge selected by the e.p. in $\vec{\mathbf{s}}$. Since G contains more than one edges, there exists an $e' \in E(G)$ not selected by the e.p. in $\vec{\mathbf{s}}$, such that e and e' contain at least one different endpoint, assume u. If there is at least one v.p. located on e, it will prefer to go to u so that not to get arrested by the edge player and gain more. Thus, this case can not be a pure NE for the vertex players. Otherwise, the edge e contains no vertex player. But in this case, the e.p. would like to change current action and select another edge, where there is at least one vertex player, so that to gain more. Thus, again this case can not be a pure NE, for the e.p. this time. Since always in any case, one of the two kinds of players is not satisfied by $\vec{\mathbf{s}}$, $\vec{\mathbf{s}}$ is not a pure NE. ■

We continue with a characterization of mixed Nash equilibria:

Theorem 3.2 *(***Characterization of Mixed NE***) A mixed strategy profile $\vec{\mathbf{s}}$ is a Nash equilibrium for any $\Pi(G)$ if and only if:*

1. *$D_{\vec{\mathbf{s}}}(ep)$ is an* edge cover *of G and $D_{\vec{\mathbf{s}}}(VP)$ is a* vertex cover *of the graph induced by $D_{\vec{\mathbf{s}}}(ep)$.*
2. *The probability distribution of the e.p. over E, is such that, (a) $P_{\vec{\mathbf{s}}}(Hit(v)) = P_{\vec{\mathbf{s}}}(Hit(u)) = \min_v P_{\vec{\mathbf{s}}}(Hit(v))$, $\forall\ u, v \in D_{\vec{\mathbf{s}}}(VP)$, (b) $P_{\vec{\mathbf{s}}}(Hit(v)) \leq P_{\vec{\mathbf{s}}}(Hit(u))$, for any $v, u \in V$, $v \in D_{\vec{\mathbf{s}}}(VP)$, $u \notin D_{\vec{\mathbf{s}}}(VP)$ and (c) $\sum_{e \in D_{\vec{\mathbf{s}}}(ep)} P_{\vec{\mathbf{s}}}(ep, e) = 1$.*
3. *The probability distributions of the vertex players over V are such that, (a) $m_{\vec{\mathbf{s}}}(e_1) = m_{\vec{\mathbf{s}}}(e_2) = \max_e m_{\vec{\mathbf{s}}}(e)$, $\forall\ e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in D_{\vec{\mathbf{s}}}(ep)$, (b) $m_{\vec{\mathbf{s}}}(e_1) \geq m_{\vec{\mathbf{s}}}(e_2)$, $\forall\ e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E$, $e_1 \in D_{\vec{\mathbf{s}}}(ep)$, $e_2 \notin D_{\vec{\mathbf{s}}}(ep)$ and (c) $\sum_{v \in V(D_{\vec{\mathbf{s}}}(ep))} m_{\vec{\mathbf{s}}}(v) = n$.*

Remark. Note that the characterization implies no polynomial time algorithm for computing a mixed Nash equilibrium (unless $\mathcal{P} = \mathcal{NP}$), since it involves solving a mixed integer programming problem.

4 Matching Nash Equilibria

We introduce a family of configurations, called *matching*, which (as we show) can lead to mixed NE for the problem considered, called *matching* mixed NE on any instance of the problem. We provide an if and only if characterization for the existence of a *matching* mixed NE. Using this characterization, we provide a polynomial time algorithm for the computation of *matching* Nash equilibria for any instance $\Pi(G)$ of the problem, where the graph G satisfies the characterization. We remark applicability of the algorithm for a quite broad family of graphs, that of *bipartite graphs* (section 5).

The obvious difficulty of solving the system of Theorem 3.2 directs us in trying to investigate the existence of some *matching*, i.e. polynomially computable, solutions of the system, corresponding to mixed NE of the game. To which configuration should we consider as *matching* or easy to compute, we utilized the following way of thinking. A first observation is that finding a configuration that satisfies condition **2** of Theorem 3.2 seems the most difficult constrain (among the three conditions) to be fulfilled. This is so because it contains the largest number of variables among the three conditions ($P_{\vec{\mathbf{s}}}(ep, e)$, $\forall\ e \in E$) and each equation of it might involve up to $\Delta(G)$ such variables. Thus, let us consider the subtask of the system of Theorem 3.2 of computing function $P_{\vec{\mathbf{s}}}(\cdot)$, $\forall e \in E$. Consider the case where the equations of condition **2.(a)** are *independent*, that is for each variable e,

$P_{\vec{s}}(ep, e)$ appears in only one equation of condition **2.(a)**. Obviously, in this case the task becomes less difficult. Note that in order these equations to be *independent* , $D_{\vec{s}}(VP)$ should constitute an independent set of G. Moreover, when each vertex of $D_{\vec{s}}(VP)$ is incident only in one edge of $D_{\vec{s}}(ep)$, then each equation of condition **2.(a)** contains only one variable, making the satisfaction of the condition even less difficult. Based on these thoughts, we define the following family of configurations which, as we show, can lead to mixed NE for the game. In the sequel, we investigate their existence and their polynomial time computation.

Definition 4.1 *A* **matching configuration** $\vec{s}$ *of* $\Pi(G)$ *satisfies:* **(1)** *$D_{\vec{s}}(VP)$ is an independent set of G and* **(2)** *each vertex v of $D_{\vec{s}}(VP)$ is incident to only one edge of $D_{\vec{s}}(ep)$.*

Claim 4.1 *For any graph G, if in $\Pi(G)$ there exists a* matching *configuration which additionally satisfies condition* **1** *of Theorem 3.2, then there exists probability distributions for the vertex players and the e.p. such that the resulting configuration is a mixed Nash equilibrium for $\Pi(G)$. Also, these distributions can be computed in polynomial time.*

Proof. Consider any configuration $\vec{s}$ as stated by the Claim (assuming that there exists one) and the following probability distributions of the vertex players and the e.p. on $\vec{s}$:

$$\mathbf{e}.p. : \forall\, e \in D_{\vec{s}}(ep),\ P_{\vec{s}}(ep, e) = 1/|D_{\vec{s}}(ep)| \ \textit{ and } \ \forall\ e' \in E,\ e' \notin D_{\vec{s}}(ep),\ P_{\vec{s}}(ep, e') = 0 \quad (3)$$

$$\mathbf{v.p_i} : \forall\, i \in \mathcal{N}_{VP},\ \forall\, v \in D_{\vec{s}}(VP),\ P_{\vec{s}}(vp_i, v) := \frac{1}{|D_{\vec{s}}(VP)|} \ \textit{ and } \ \forall\, u \in V,\ u \notin D_{\vec{s}}(VP),\ P_{\vec{s}}(vp_i, u) = 0 \quad (4)$$

Proposition 4.2

$$\forall\, v \in D_{\vec{s}}(VP),\ \ m_{\vec{s}}(v) = \frac{n}{|D_{\vec{s}}(VP)|} \ \textit{ and } \ \forall\, u \in V,\ u \notin D_{\vec{s}}(VP),\ m_{\vec{s}}(u) = 0$$

Proof.

$$\forall\, v \in D_{\vec{s}}(VP),\ \ m_{\vec{s}}(v) = \sum_{i \in \mathcal{N}_{VP}} P_{\vec{s}}(vp_i, v) = \sum_{i \in \mathcal{N}_{VP}} \frac{1}{|D_{\vec{s}}(VP)|} = \frac{n}{|D_{\vec{s}}(VP)|}$$

by equation (4). In contrast, for any other vertex $u \in V$, $u \notin D_{\vec{s}}(VP)$, by the same equation, $m_{\vec{s}}(u) = 0$. ∎

We show that $\vec{s}$ satisfies all conditions of Theorem 3.2, thus it is a mixed NE.

1.: By assumption.

2.(a): $P_{\vec{s}}(Hit(v)) = \frac{1}{|D_{\vec{s}}(ep)|}$, $\forall\ v \in D_{\vec{s}}(VP)$, by condition (2) of the definition of a *matching* configuration and equation (3) above.

2.(b): $P_{\vec{s}}(Hit(u)) \geq \frac{1}{|D_{\vec{s}}(ep)|} = P_{\vec{s}}(Hit(v))$, $\forall\, u \notin D_{\vec{s}}(VP)$ and $v \in D_{\vec{s}}(VP)$, because $D_{\vec{s}}(ep)$ is an edge cover of G (by assumption), recalling condition **2(a)** proved above and equation (3) above.

2.(c): $\sum_{e \in D_{\vec{s}}(ep)} \ P_{\vec{s}}(ep, e) = \sum_{e \in D_{\vec{s}}(ep)} \frac{1}{|D_{\vec{s}}(ep)|} = 1$.

3.(a): $m_{\vec{s}}(e_1) = m_{\vec{s}}(v_1) + m_{\vec{s}}(u_1) = 0 + \frac{n}{|D_{\vec{s}}(VP)|} = \frac{n}{|D_{\vec{s}}(VP)|}$, $\forall\, e_1 = (u_1, v_1) \in D_{\vec{s}}(ep)$, because D_{ep} is an edge cover of G (by assumption), D_{vp} is an independent set of G (condition (1) of the definition of a *matching* configuration) and recalling Proposition 4.2 above.

3.(b): $m_{\vec{s}}(e_2) = m_{\vec{s}}(v_2) + m_{\vec{s}}(u_2) = 0 + \frac{n}{|D_{\vec{s}}(VP)|}$, $\forall\, e_2 = (v_2, u_2) \in E, \notin D_{\vec{s}}(ep)$, using the same arguments as in **3.(a)**. Combining this with the condition **3.(a)** proved above, we get that $m_{\vec{s}}(e_1) = m_{\vec{s}}(e_2)$, $\forall\, e_1 = (u_1, v_1) \in D_{\vec{s}}(ep)$, $\ e_2 = (u_2, v_2) \in E,\ e_2 \notin D_{\vec{s}}(ep)$.

3.(c): Since $D_{evp}(\vec{s})$ is an edge cover of G (by assumption) and by Proposition 4.2, we have $\sum_{v \in V(D_{\vec{s}}(ep))} m_{\vec{s}}(v) = \sum_{v \in V} \frac{n}{|D_{\vec{s}}(VP)|} = |D_{\vec{s}}(VP)| \cdot \frac{n}{|D_{\vec{s}}(VP)|} = n$.

Note that the probability distributions for the vertex players and the e.p. can be computed in polynomial time. ∎

Definition 4.2 *A* matching *configuration which additionally satisfies condition* **1** *of Theorem 3.2 is called a* **matching mixed NE**.

We proceed to characterize graphs that admit *matching* Nash equilibria.

Theorem 4.3 *For any graph G, $\Pi(G)$ contains a* matching *mixed Nash equilibrium if and only if the vertices of the graph G can be partitioned into two sets IS, VC ($VC \cup IS = V$ and $VC \cap IS = \emptyset$), such that IS is an independent set of G (equivalently, VC is a vertex cover of the graph) and G is a VC-expander graph.*

Proof. *We first prove that if G has an independent set IS and the graph G is a VC-expander graph, where $VC = V \backslash IS$, then $\Pi(G)$ contains a* matching *mixed NE.* By the definition of a VC-expander graph, it holds that $N(VC') \geq VC'$, for all $VC' \subseteq VC$. Thus, by Hall's Theorem 2.1, G has a matching M such that each vertex of VC is matched into a vertex of $V \backslash VC$ in M. Now, since M is a matching, for any vertex $u \in VC$, $\exists\, e = (u, v) \in M$, $v \in V \backslash VC = IS$. Partition IS into two sets IS_1, IS_2, where set IS_1 consists of vertices $v \in IS$ for which there exists an $e = (u, v) \in M$, $u \in VC$. Let IS_2 the remaining vertices of the set, i.e. $IS_2 = IS \backslash IS_1$ and $\forall\, v \in IS_2 : \nexists\, (u, v) \in M$, $u \in VC$.

Now, recall that there is no edge between any two vertices of set IS, since it is independent set, by assumption. Henceforth, since G is a connected graph, $\forall\; u \in IS_2 \subseteq IS$, there exists $e = (u, v) \in E$. Note also that for each such edge $e = (u, v)$, $v \in V \backslash IS = VC$. Now, construct set M consisting of all those edges as follows: set initially $M := \emptyset$ and $\forall u \in IS_2$, iteratively set $M := M \cup (u, v)$, for any $(u, v) \in E$, $v \in VC$. Note that each vertex $v \in VC$ is considered only once in the construction of set M. Note that $M_1 \cap M = \emptyset$. We construct the following configuration $\vec{\mathbf{s}}$ of $\Pi(G)$: Set $D_{\vec{\mathbf{s}}}(VP) = IS$ and $D_{\vec{\mathbf{s}}}(ep) = M \cup M_1$.

We first show that that $\vec{\mathbf{s}}$ is a *matching* configuration. Condition (1) is fulfilled because IS is an independent set. We show that condition (2) of a *matching* configuration is fulfilled. Each vertex of set IS belongs either to IS_1 or to IS_2. By definition, each vertex of IS_1 is incident to only one edge of M and each vertex of IS_2 is incident to no edge in M and is incident to only one edge of M_1. Thus, the condition (2) of a *matching* configuration holds.

We next show that condition **1** of Theorem 3.2 is satisfied by $\vec{\mathbf{s}}$. We first show that $D_{\vec{\mathbf{s}}}(ep)$ is an edge cover of G. This is true because (i) set $M \subseteq D_{\vec{\mathbf{s}}}(ep)$ covers all vertices of set VC and IS_1, by its construction and (ii) set $M_1 \subseteq D_{\vec{\mathbf{s}}}(ep)$ covers all vertices of set IS_2 which are the remaining vertices of G not covered by set M, also by its construction. Thus, $D_{\vec{\mathbf{s}}}(ep)$ is an edge cover of the graph G. We next show that $D_{\vec{\mathbf{s}}}(VP)$ is a vertex cover of the subgraph of G induced by set $D_{\vec{\mathbf{s}}}(ep)$. By the definition of sets IS_1, $IS_2 \subseteq IS$, any edge $e \in M$ is covered by a vertex of set IS_1 and each edge $e \in M_1$ is covered by a vertex of set IS_2. Henceforth, each edge of set $D_{\vec{\mathbf{s}}}(ep)$ is covered by set IS, henceforth the set is a vertex cover of the graph induced by $D_{\vec{\mathbf{s}}}(ep)$. We conclude that condition **1** of Theorem 3.2 is satisfied by $\vec{\mathbf{s}}$. Henceforth, by Claim 4.1, it can lead to a *matching* mixed NE of $\Pi(G)$.

We proceed to show that if G contains a matching *mixed NE, assume $\vec{\mathbf{s}}$, then G has an independent set IS and the graph G is a VC-expander graph, where $VC = V \backslash IS$.* Define sets $IS = D_{\vec{\mathbf{s}}}(VP)$ and $VC = V \backslash IS$. We show that these sets satisfy the above requirements. Note first that, set IS is an independent of G since $D_{\vec{\mathbf{s}}}(VP)$ is an independent set of G by condition (1) of the definition of a *matching* configuration.

We next show G contains a matching M such that each vertex of set VC is matched into a vertex of $V \backslash VC$ in M. Henceforth, by Hall's Theorem 2.1, we get that $N(VC') \geq VC'$, for all $VC' \subseteq VC$ and so G is a VC-expander. Since $D_{\vec{\mathbf{s}}}(ep)$ is an edge cover of G (condition **1** of a mixed NE of Theorem 3.2), for each $v \in VC$, $\exists\, e = (u, v) \in D_{\vec{\mathbf{s}}}(ep)$. Note that, for this edge ($e = (u, v)$), $v \in IS$, since otherwise IS would not be a vertex cover of $D_{\vec{\mathbf{s}}}(ep)$ (Condition **1** of a mixed NE). Moreover, by condition (2) of a *matching* mixed NE, any vertex of IS has only one incident to it edge in

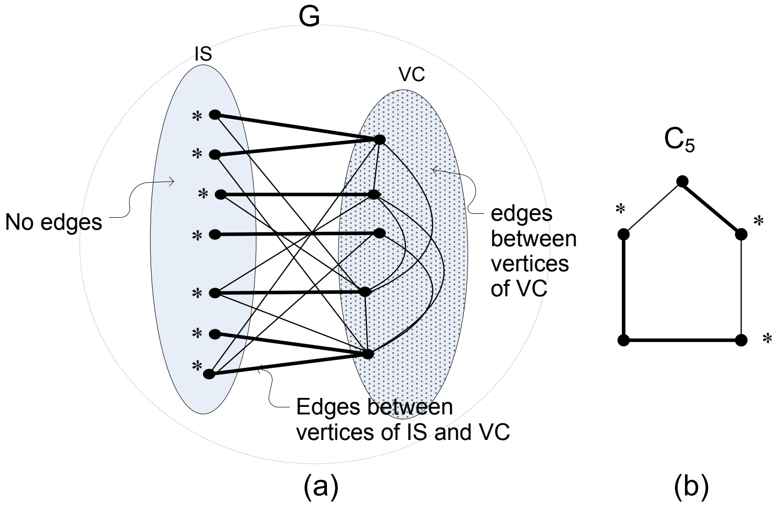

Figure 1: Examples of graphs (a) with and (b) without *matching* mixed Nash equilibrium.

$D_{\vec{\mathbf{s}}}(ep)$. Thus, we conclude that for each $v \in VC$, vertex $u \in IS$, such that $e = (u, v) \in D_{\vec{\mathbf{s}}}(ep)$, has no other edge than e incident to it in $D_{\vec{\mathbf{s}}}(ep)$. Now, construct set M consisting of all those edges as follows: set initially $M := \emptyset$ and $\forall u \in IS_2$, iteratively set $M := M \cup (u, v)$, for any $(u, v) \in E$, $v \in VC$. Note that each vertex $v \in VC$ is considered only once in the construction of set M. Thus, set M is a matching and moreover each vertex of set VC is matched into a vertex of $VC \backslash V$ in M, by its construction. Summing up, we get that IS is an independent set of G and the graph G is a VC-expander graph, where $VC = V \backslash IS$, as required. ∎

An example of a graph G with a *matching* mixed NE $\vec{\mathbf{s}}$ is illustrated in Figure 1. Set $D_{\vec{\mathbf{s}}}(ep)$ is denoted by bold edges and set $D_{\vec{\mathbf{s}}}(VP)(= IS)$ (as in Theorem 4.3) by vertices with an asterisk, $*$. We remark that *not* all graphs G have a *matching* mixed NE; any odd cycle is such graph; this is so because for every edge cover EC of the graph (corresponding to $D_{\vec{\mathbf{s}}}(ep)$), there is no set $VC \subseteq V$ (corresponding to $D_{\vec{\mathbf{s}}}(VP)$) such that VC is a vertex cover of the graph induced by EC and VC is also an independent set of G. See Figure 1(b) for an example.

4.1 A Polynomial Time Algorithm

The previous Theorems and Lemmas enables us to develop a polynomial time algorithm for finding *matching* mixed NE for any $\Pi(G)$, where G is a graph satisfying the requirements of Theorem 4.3. The Algorithm is described in pseudocode in Figure 2.

Theorem 4.4 *(***Correctness***) Algorithm A computes a* matching *mixed Nash equilibrium for* $\Pi(G)$.

Proof.

Claim 4.5 *Step* **1** *of Algorithm A terminates successfully in finite number of steps.*

Proof. Consider any iteration of step **1.(b)**, which is the only non-trivial step of the procedure. We prove that step **1.(b)ii.** is always feasible as far as $Unmatched \neq \emptyset$. Since in every iteration of step **1.(b)**, set *Unmatched* is decreased by one, the loop of the step terminates in finite number of iterations. The statement is proved by induction.

We first prove that initially step **1.(b)ii.** is feasible. Note that initially, $Unmatched = VC$. Recall since G is a VC-expander graph, it holds that $|N_G(S)| \geq |S|$, for all $S \subseteq VC$. Thus, by Hall's Theorem 2.1, $\exists\ v \in V \backslash Unmatched$ such that $(u, v) \in E$. Henceforth, step **1.(b)ii.** is initially feasible.

Algorithm $A(\Pi(G), IS, VC)$

INPUT: A game $\Pi(G)$ and a partition of $V(G)$ into sets IS, $VC = V \backslash IS$, such that IS is an independent set of G and G is a VC-expander graph.
OUTPUT: A mixed NE $\vec{\mathbf{s}}$ for $\Pi(G)$.

1. Compute a set $M \subseteq E$, as follows:
   (a) *Initialization*: Set $M := \emptyset$, $Matched := \emptyset$ (currently matched vertices in M), $Unmatched := VC$ (currently unmatched vertices in M vertices of VC), $Unused := IS$, $i := 1$, $G_i := G$ and $M_1 := \emptyset$.
   (b) While $Unmatched \neq \emptyset$ Do:
      i. Consider a $u \in Unmatched$ and compute $N_{G_i}(u)$.
      ii. Find a $v \in Unused$ such that $(u, v) \in E_i$. Set $M := M \cup (u, v)$, $Unused := Unused \backslash \{v\}$.
      iii. *Prepare next iteration*: Set $i := i + 1$, $Matched := Matched \cup \{u\}$, $Unmatched := Unmatched \backslash \{u\}$, $G_i := G_{i-1} \backslash u \backslash v$.
2. Partition set IS into two sets IS_1, IS_2 as follows: $IS_1 := \{u \in IS : \exists\ (u, v) \in M\}$ and $IS_2 := IS \backslash IS_1$. Note that $IS_2 := \{u \in IS : \nexists\ (u, v) \in M,\ v \in VC\}$.
   Compute a set M_1 as follows: $\forall\ u \in IS_2$, set $M_1 := M_1 \cup (u, v)$, for any $(u, v) \in E$, $v \in VC$.
3. Define a configuration $\vec{\mathbf{s}}$ with the following support: $D_{\vec{\mathbf{s}}}(VP) := IS$, $D_{\vec{\mathbf{s}}}(ep) := M \cup M_1$.
4. Determine the probabilities distributions of the vertex players and the e.p. of configuration $\vec{\mathbf{s}}'$ using equations (3) and (4) of Claim 4.1.

Figure 2: Algorithm A.

Consider any iteration $i > 1$ of step **1.(b)ii.** of the algorithm. Denote as $Unmatched_i$, $Matched_i$, $Unused_i$ and M_i to be sets $Unmatched$, $Matched$, $Unused$ and M respectively, of iteration i of the step. Let also vertex $u_i \in Unmatched_i$ and $v_i \in Unused_i$, the vertices added to set $Matched$ and removed from set $Unused$, respectively, of iteration i of the step. Assume, by induction, $|N_{G_i}(Unm)| \geq |Unm|$, for all $Unm \subseteq Unmatched_i$. Thus, by Hall's Theorem 2.1, $\exists\ v_i \in V_i \backslash Unmatched_i = Unused_i$ such that $(u_i, v_i) \in E_i$. Henceforth, step **1.(b)ii.** is feasible.

We prove that in the next iteration, $i+1$, of step **1.(b)ii.**, if $Unmatched \neq \emptyset$, then $|N_{G_{i+1}}(Unm)| \geq |Unm|$, for all $Unm \subseteq Unmatched_{i+1}$. Thus, by Hall's Theorem 2.1, $\exists v_{i+1} \in V_{i+1} \backslash Unmatched_{i+1} = Unused_{i+1}$ such that $(u_{i+1}, v_{i+1}) \in E_{i+1}$ and henceforth iteration $i + 1$ of step **1.(b)ii.** would be feasible. Note that $Unmatched_{i+1} = Unmatched_i \backslash \{u_i\}$ and $N_{G_{i+1}}(Unmatched_{i+1}) = N_{G_i}(Unmatched_i) \backslash \{v_i\}$. Since, $|N_{G_i}(Unm)| \geq |Unm|$, for all $Unm \subseteq Unmatched_i$, we get that $|N_{G_{i+1}}(Unm')| \geq |Unm'|$, for all $Unm' = Unm \backslash \{u_i\} \subseteq Unmatched_{i+1}$.

■

Finally, observe that set M computed by step **1** of the algorithm is a matching of G. This is true because each vertex of VC is considered only once in the loop and each vertex $v \in IS$ contributes only to one edge (u, v) of set M and is then removed from G, by operation $Unused := Unused \backslash \{v\}$ of step **1.(b)ii.**. The successful termination of step **1.** proved in Claim 4.5 guarantees that in the constructed matching M, each vertex of set VC is matched into a vertex of IS in M.

Note that sets IS, M are the same as their corresponding sets in the proof of Theorem 4.3. Furthermore, note the assignment for support of configuration $\vec{\mathbf{s}}$ of step **3** of the algorithm, is equivalent to that of the configuration of Theorem 4.3. This, combined with the above observations (on sets IS, M and M_1 involved in the assignment) concludes that the configuration of Theorem

4.3 has the same support as configuration $\vec{\mathbf{s}}$ of the algorithm.

Now, using the same arguments as in Theorem 4.3, we can prove that configuration $\vec{\mathbf{s}}$ constructed here, is a *matching* configuration and also that condition **1** of a mixed NE of Theorem 3.2 is satisfied in $\vec{\mathbf{s}}$. Moreover, note that the probability distributions of the vertex players and the e.p. of configuration $\vec{\mathbf{s}}$ here is the same as that of Claim 4.1. Henceforth, $\vec{\mathbf{s}}$ is *matching* mixed NE of Π. ∎

Theorem 4.6 *(***Time Complexity***) Algorithm A terminates in linear time $O(\mu)$.*

Proof. **Step 1**: In any graph $\beta(G) \leq \nu$. Thus, step **1.(b)** is iterated at most ν times. Since any edge vertex of G has a degree at most $\nu - 1$, steps **1.(b) i.** and **ii.** can be accomplished in $O(\nu)$ time. Thus, step **1** of the algorithm is finished in the same time ($O(\nu)$). **Steps 2, 3** and **4**: They take $O(\nu)$, $O(\mu)$ and $O(\mu)$ time, respectively. Summing up, we conclude that Algorithm A needs $O(\mu)$ to be accomplished. ∎

5 Bipartite Graphs

In this section we investigate the existence and polynomial time computation of *matching* mixed Nash equilibria for any $\Pi(G)$, for which G is a bipartite graph. We first provide some useful Lemmas and Theorems on important properties of bipartite graphs. Using them, we prove that in any bipartite graph G, $\Pi(G)$ always contains a *matching* mixed NE. Using these results, in the sequel, we show that Algorithm A (section 4) can apply on bipartite graphs, providing a polynomial time algorithm for computing such an equilibrium on $\Pi(G)$.

5.1 Existence

Lemma 5.1 *In any bipartite graph G there exists a matching M and a vertex cover VC such that* **(1)** *every edge in M contains exactly one vertex of VC and* **(2)** *every vertex in VC is contained in exactly one edge of M.*

Proof. Let X, Y the bipartition of the bipartite graph G. Consider any minimum vertex cover of the graph G, VC. We are going to construct a matching M of G so that conditions (1) and (2) of the Lemma hold. Let R the vertices of VC contained in set X, i.e. $R = VC \cap X$ and T the vertices of VC contained in set Y, i.e. $T = VC \cap Y$. Note that $VC = R \cup T$. Let H and H' the subgraphs of G induced by $R \cup (Y - T)$ and $T \cup (X - R)$, respectively. We are going to show that G contains a matching in M as required by the Lemma.

Since $R \cup T$ is a vertex cover, G has no edge from $Y - T$ to $X - R$. We show that for each $S \subseteq R$, $N_H(S) \subseteq Y - T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in VC to obtain a smallest vertex cover ($*1$). This is true because (i) $N_H(S)$ covers all edges incident to S that are not covered by T and (ii) since G is a bipartite graph there are no edges between the vertices of set S, so that a possible substitute of set S do not need to cover any such edge.

Thus, $|N_H(S)| \geq |S|$, for all $S \subseteq R$. By Hall's Theorem (Theorem 2.1), H has a matching M_H such that each vertex of R is matched in M_H. Using similar arguments for set T, we can prove that for each $S' \subseteq T$, $|N_{H'}(S')| \geq |S'|$. Henceforth, H' has a matching $M_{H'}$ such that each vertex of T is matched in $M_{H'}$. Now define $M = M_H \cup M_{H'}$. Since each H, H' is an induced subgraph of G and the two subgraphs have disjoint sets of vertices, we get that M is matching of G and that each vertex of $VC = R \cup T$ is matched in M. This result combined with the fact that M is a matching of G concludes that condition (2) of the Lemma holds.

We proceed to prove condition (1). That is, to show that every edge of M contains exactly one vertex of VC. Observe first that by the construction of set M, every edge of M contains at least one vertex of VC. Moreover, note that only one of the endpoints of the edge is contained in M. This is true because by the construction of set M each edge of set M matches either (i) a vertex of set $R \subseteq X$ to a vertex of set $(Y - T) \subseteq Y$ or (ii) a vertex of set $T \subseteq Y$ to a vertex of set $(X - R) \subseteq X$. So, for any case exactly one of the two endpoints of the edge is not contained in VC. ■

Remark. The statement of the Lemma does not hold for all graphs; any odd cycle graph is an example of its falseness (See Figure 1(b)). The falseness of the Lemma in a general graph consists in that the statement ($*1$) in its proof is false; condition (ii) required (for proving $*1$) is not true.

Lemma 5.2 *Any X, Y-bigraph graph G can be partitioned into two sets IS, VC ($IS \cup VC = V$ and $IS \cap VC = \emptyset$) such that VC is a vertex cover of G (equivalently, IS is an independent set of G) and G is a VC-expander graph.*

Proof. Let set VC of the Theorem to be a minimum vertex cover of G, as in Lemma 5.1. Consider also a matching M of G from set VC to set $V \backslash VC = IS$ as in Lemma 5.1. Note that, $IS = V \backslash VC$ is an independent set of G. By Lemma 5.1, each vertex $v \in VC$ is matched into a vertex of IS in M, that is, $\forall\, u \in VC$, $\exists\, e = (u, v) \in M$, $v \in IS$. Thus, by Hall's Theorem 2.1, G is a VC-expander graph. Thus, sets IS and VC satisfy the requirements of the Theorem. ■

Lemma 5.2 and Theorem 4.3 finally imply:

Theorem 5.3 *Any $\Pi(G)$ for which G is a connected bipartite graph, contains a* matching *mixed Nash equilibrium.* ■

5.2 Computation

Theorem 5.4 *For any $\Pi(G)$, for which G is a bipartite graph, a* matching *mixed Nash equilibrium of $\Pi(G)$ can be computed in polynomial time, $\max\{O(\mu\sqrt{\nu}), O(\nu^{2.5}/\sqrt{\log \nu})\}$, using Algorithm A.*

Proof. We consider any set VC, as described Lemma 5.2, i.e. any minimum vertex cover of G and compute also set $V \backslash VC = IS$.

Proposition 5.5 *A minimum vertex cover of a bipartite graph can be computed in polynomial time, $\max\{O(\mu\sqrt{\nu}), O(\nu^{2.5}/\sqrt{\log \nu})\}$.*

Proof. Recall König's Theorem ([5], or [2]), stating that for any bipartite graph G, the maximum size of a matching M in G is equal to the minimum size of a vertex cover VC of G. Actually the Theorem suggests a polynomial time algorithm for computing a minimum vertex cover VC of G, assuming that a maximum matching M of G is given (see also [2], Theorem 10.2.1, page 180). Such an algorithm takes $O(\nu^{2.5}/\sqrt{\log \nu})$ time.

Thus, one can compute a maximum matching M of a bipartite graph G in polynomial time ($O(\mu\sqrt{\nu})$), using the algorithm of [3]. By the above observations on König's Theorem ([5]), from M we can compute a minimum vertex cover VC of the graph in additional time of $O(\nu^{2.5}/\sqrt{\log \nu})$. Thus, the whole procedure of computing a minimum vertex cover of a bipartite graph G needs $\max\{O(\mu\sqrt{\nu}), O(\nu^{2.5}/\sqrt{\log \nu})\}$ time. ■

Thus, we can apply Algorithm A on $\Pi(G)$ using those sets IS, VC, as input. By Theorem 4.6, we get that the whole procedure takes time $O(\mu) + \max\{O(\mu\sqrt{\nu}), O(\nu^{2.5}/\sqrt{\log \nu})\} = \max\{O(\mu\sqrt{\nu}), O(\nu^{2.5}/\sqrt{\log \nu})\}$.

■

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APPENDIX

Proof of Theorem 3.2.

Claim 3.2.1 *In any mixed NE* $\vec{\mathbf{s}}^*$*,* $D_{ep}(\vec{\mathbf{s}}^*)$ *is an edge cover of* G *of instance* $\Pi(G)$*.*

Proof. Assume the contrary. Let NC a set of vertices of G not *covered* by $D_{ep}(\vec{\mathbf{s}}^*)$. Then, $\forall\ i \in \mathcal{N}_{VP}$, $D_{vp_i}(\vec{\mathbf{s}}^*) \subseteq NC$ because these actions give to the v.p. zero probability to be catched by the e.p.. But in such case, the e.p. would gain nothing, because there will be no v.p. on its support, while it could select any other edge, where there is at least one v.p. (obviously such an edge exists, because the vertex players have to be somewhere on V) and gain more. Thus, this strategy profile is not a mixed NE, a contradiction. Henceforth, the initial assumption is false. ■

Claim 3.2.2 *In any mixed NE* $\vec{\mathbf{s}}^*$*,* $D_{vp}(\vec{\mathbf{s}}^*)$ *is a vertex cover of the graph obtained by* $D_{ep}(\vec{\mathbf{s}}^*)$*.*

Proof. Assume the contrary. Let $e = (u, v) \in D_{ep}(\vec{\mathbf{s}}^*)$ an edge not *covered* by $D_{vp}(\vec{\mathbf{s}}^*)$. Then, the gain of the e.p. on e would be zero (since there is no v.p. on it). Thus, the e.p. should gain more if moving the probability of choosing edge e to another edge for which at least one of its endpoints are *covered* by $D_{vp}(\vec{\mathbf{s}}^*)$ (obviously such an edge exists, because the vertex players have to be somewhere on V). Thus, this strategy profile can not be a mixed NE, a contradiction, henceforth the initial assumption is false. ■

Claim 3.2.3 *In any mixed strategy profile* $\vec{\mathbf{s}}^*$ *of* $\Pi(G)$*,* $\sum_{v \in V(D_{\vec{\mathbf{s}}*}(ep))} m_{\vec{\mathbf{s}}}(v) = n$*.*

Proof.

$$\begin{aligned}
\sum_{v \in V(D_{\vec{\mathbf{s}}*}(ep))} m_{\vec{\mathbf{s}}*}(v) &= \sum_{v \in V(D_{\vec{\mathbf{s}}*}(ep))} \sum_{i \in \mathcal{N}_{VP}} P_{\vec{\mathbf{s}}}(vp_i, v) \ \Leftrightarrow \\
&= \sum_{i \in \mathcal{N}_{VP}} \sum_{v \in V(D_{\vec{\mathbf{s}}*}(ep))} P_{\vec{\mathbf{s}}*}(vp_i, v) = \sum_{i \in \mathcal{N}_{VP}} \sum_{v \in V} P_{\vec{\mathbf{s}}*}(vp_i, v) \ (\textit{by Claim } 3.2.1) \\
&= \sum_{v \in V} \sum_{i \in \mathcal{N}_{VP}} P_{\vec{\mathbf{s}}*}(vp_i, v) = \sum_{i \in \mathcal{N}_{VP}} (1) = |\mathcal{N}_{VP}| = n
\end{aligned}$$

■

Next we prove that if $\vec{\mathbf{s}}$ is a mixed NE for $\Pi(G)$ then conditions 1-3 hold. **1.**: By Claims 3.2.1, 3.2.2. **2.(a)**: By eq. (1) and the basic game theory on Nash equilibria, for any two $u, v \in D_{\vec{\mathbf{s}}}(VP)$, and any $i \in N_{VP}$, $\mathsf{IC}_i = 1 - P_{\vec{\mathbf{s}}}(Hit(v)) = 1 - P(hid_u, \vec{\mathbf{s}})$. **2.(b)**: Again by basic game theory on Nash equilibria, for any $i \in N_{VP}$ and any $v \in D_{\vec{\mathbf{s}}}(VP)$, $u \notin D_{\vec{\mathbf{s}}}(VP)$ by eq. (1), $1 - P_{\vec{\mathbf{s}}}(Hit(v)) \geq 1 - P(hid_u, \vec{\mathbf{s}})$ and henceforth, $P_{\vec{\mathbf{s}}}(Hit(v)) \leq P(hid_u, \vec{\mathbf{s}})$. **2.(c)**: obvious since $P_{\vec{\mathbf{s}}}(ep)$ is a probability distribution over E. **3.(a) and (b)**: by eq. (2) and basic game theory on Nash equilibria.

Next, we prove that if conditions 1-3 hold for a mixed strategy profile $\vec{\mathbf{s}}$ of $\Pi(G)$ then $\vec{\mathbf{s}}$ is a NE. Consider first the vertex players. For any vp_i, by 2.(a), any $v \in D_{vp_i}(\vec{\mathbf{s}})$, has minimum hitting probability in $\vec{\mathbf{s}}$. Thus, any of these vertices is a *best response* ([8], chapter 3) choice for the vp_i. in $\vec{\mathbf{s}}$. Thus, vp_i is satisfied in $\vec{\mathbf{s}}$. Next, consider the e.p.. By 3.(a) and 3.(b), any $e \in D_{\vec{\mathbf{s}}}(ep)$, has a maximum expected number of vertex players on it. Thus, e is a best response choice for ep in $\vec{\mathbf{s}}$ and henceforth e is satisfied in $\vec{\mathbf{s}}$. ■