Radiocolorings in Periodic Planar Graphs: PSPACE-Completeness and Efficient Approximations for the Optimal Range of Frequencies

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Abstract

The Frequency Assignment Problem (FAP) in radio networks is the problem of assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modelled by variations of the graph coloring problem. A Radiocoloring (RC) of a graph $G(V, E)$ is an assignment function $\Lambda : V \rightarrow \mathbb{N}$ such that $|\Lambda(u) - \Lambda(v)| \geq 2$, when u, v are neighbors in G, and $|\Lambda(u) - \Lambda(v)| \geq 1$ when the distance of u, v in G is two. The discrete number of frequencies used is called *order* and the range of frequencies used, *span*. The optimization versions of the Radiocoloring Problem (RCP) are to minimize the span (*min span RCP*) or the order (*min order RCP*).

In this paper, we deal with an interesting, yet not examined until now, variation of the radiocoloring problem: that of satisfying frequency assignment requests which exhibit some *periodic* behavior. In this case, the interference graph (modelling interference between transmitters) is some (infinite) periodic graph. Infinite periodic graphs usually model finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. Alternatively, they can model very large networks produced by the repetition of a small graph.

A *periodic graph* G is defined by an infinite two-way sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all G_i. We focus on planar periodic graphs, because in many cases real networks are planar and also because of their independent mathematical interest.

We give two basic results:

- We prove that *the min span RCP is PSPACE-complete for periodic planar graphs.*

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- We provide an $O(n(\Delta(G_i) + \sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting each G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to* $\frac{5}{3}$ *as* $\Delta(G_i) + \sigma$ *tends to infinity.*

We remark that, any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α and where $\Delta(G)$ is the maximum degree of G, can be used as a subroutine in our algorithm to produce an approximation for min span RCP of asymptotic ratio α for periodic planar graphs.

Key words: approximation algorithms, computational complexity, radio networks, frequency assignment, coloring, periodic graphs.

1 Introduction, Previous Work and our Results

1.1 The Radiocoloring Problem

The Frequency Assignment Problem (FAP) in radio networks is a well-studied, interesting and well motivated problem, aiming at assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modeled by variations of the graph coloring problem. The interference between transmitters is usually modelled by the interference graph $G(V, E)$, where the set V corresponds to the set of transmitters and E represents distance constraints. The set of colors represents the available frequencies. In addition, the color of each vertex in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in the interference graph G (frequency-distance constraints). We here study an important variation of FAP, called the Radiocoloring Problem (RCP).

Consider a graph $G(V, E)$. Let $d(u, v)$ is the distance between u and v in G, $\Delta(G)$ the maximum degree of the graph G and $n = |V|$.

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Definition 1 Radiocoloring Problem(RCP) *(8): Given a graph $G(V, E)$, a Radiocoloring (RC) is a function $\Lambda : V \rightarrow N^*$ assigning integers (colors) to the vertices of G such that $|\Lambda(u) - \Lambda(v)| \geq 2$ if $d(u, v) = 1$ and $|\Lambda(u) - \Lambda(v)| \geq 1$ if $d(u, v) = 2$. The problem of finding such an assignment is called* the Radiocoloring Problem (RCP).

Two important parameters of a radiocoloring are the following:

Definition 2 order*: The number of distinct colors used in a radiocoloring assignment Λ of G is called the* order *of the assignment Λ.*

Definition 3 span*: The number $\nu = max_{v \in V} \Lambda(v) - min_{u \in V} \Lambda(u) + 1$ used in a radiocoloring assignment Λ is called the* span *of the assignment Λ.*

The optimization versions of the RCP corresponding to these parameters are the following:

Definition 4 min span RCP*: Given a graph G, find a radiocoloring assignment of G of minimum span, denoted by $\lambda_{span}(G)$.*

Definition 5 min order RCP*: Given a graph G, find a radiocoloring assignment of G of minimum order, denoted by $\lambda_{order}(G)$.*

Note that both versions of the RCP have been proved to be NP-hard even for planar and other restricted families of graphs ((9; 23; 3)).

1.2 The Periodic Planar Radiocoloring Problem

In this work we investigate the radiocoloring problem for an interesting family of infinite planar graphs, called *periodic planar graphs*. A periodic graph G is defined by an infinite sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all iterations. We call this problem the *periodic planar radiocoloring problem*.

Infinite periodic graphs usually represent finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. We note that periodic interference graphs usually represent networks of great practical interest, since in many networks the requests for frequency assignment exhibit some periodic behavior. That is, the network accepts periodic (e.g. daily) requests for frequency assignment. Each request has a starting and ending time and a node where it is applied. Two requests interfere if they apply for nearby nodes and their time intervals overlap. The assignment should be such that there is no time overlap between any two nearby requests of the same or the preceding and following periods of requests. Alternatively, infinite periodic graphs can model very large networks produced by the repetition of a small graph. Note in this context that many real networks consist of the repetition of the same component. We focus on *planar* periodic graphs, because in many cases real networks are planar and because of the independent mathematical interest of this family of graphs.

Definition 6 Linear Periodic Planar Graph G**:** *A linear periodic planar graph is defined as follows:*

Let $\widetilde{G}$ *be an arbitrary finite connected planar graph. Let* V *the vertex set of* $\widetilde{G}$*. Let also* E_0 *be the edge set of* $\widetilde{G}$*. Let* E_+ *be a specific set of ordered pairs* (u, v) *of the nodes of* $\widetilde{G}$*. Note that* E_+ *must be a set of ordered pairs of vertices whose connection according to the rule* **(c2)** *below leads to planarity preservation.*

Consider the two-way infinite sequence of graphs $\ldots, G_i, G_{i+1}, \ldots$*, where each* G_i *is isomorphic to* $\widetilde{G}$*. The infinite graph* G *is obtained from this sequence as follows:*

(a) *We assume a line (in fact, any 1-dimensional infinite simple curve) on which we select discrete points* $\ldots, i, i+1, i+2, \ldots$*, such that:*
(a1) *Each point in the line is replaced by* $\widetilde{G}$*.*
(a2) *Each edge* $(i, i+1)$ *in the line is replaced by* E_+*.*
(a3) *For any finite subset of consecutive points in the line, replacing the points of the line by graphs* $\widetilde{G}$ *end the edges between them by* E_+*, the resulting graph is planar.*
(b) *The vertex set of* G *is the union of the vertex sets of the sequence* $\ldots, G_i, G_{i+1}, \ldots$*.*
(c) *The edges of* G *are* **(c1)** *The edge set of each* G_i *(i.e., the edge set* E_0 *of* $\widetilde{G}$*)* **(c2)** *For each pair of adjacent copies of* $\widetilde{G}$*, call them* G_i*,* G_{i+1}*, we use the* E_+ *specification of* G *to connect the nodes of* G_i *corresponding to the first elements of the pairs in* E_+ *to the nodes of* G_{i+1}*, corresponding to the second elements of the pairs in* E_+*.*

We denote a linear periodic planar graph by $G = (\widetilde{G}(V, E_0), E_+)$*.*

We note the similarity of linear periodic graphs to 1-dimensional periodic graphs defined in the work of (17).

Definition 7 $(\widetilde{G}, E_+)$*: The pair* $(\widetilde{G}, E_+)$ *is called the finite specification of* G*.*

Note that there can be infinite periodic graphs which are not linear; consider the periodic graph whose graph $\widetilde{G}$ in its finite specification, is a cycle connecting vertices a, b, c, d, and $E_+ = \{(a, a), (b, b), (c, c), (d, d)\}$, illustrated in Figure 1. This graph is not a linear periodic planar graph, because there is no line of discrete points such that (i) each point in the line can be replaced by $\widetilde{G}$ and (ii) each of its edges can be replaced by E_+ leading to planarity.

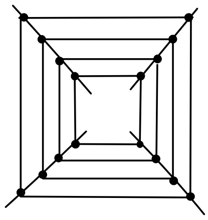

Fig. 1. This graph is not a linear periodic planar graph.

Note 1 *All our results refer to linear periodic planar graphs, which we call periodic planar graphs in the sequel.*

1.3 Our results

In this work we provide the following results:

1. We first prove that *the min span radiocoloring problem is $PSPACE$-complete for periodic planar graphs.* (The space is polynomial with respect to the size of the finite specification ($\widetilde{G}$, E_+.)
2. We provide an $O(n(\Delta(G_i) + \sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to $\frac{5}{3}$ as $\Delta(G_i) + \sigma$ tends to infinity.*

   We remark that, any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α and where $\Delta(G)$ is the maximum degree of G, can be used as a subroutine in our algorithm to produce an approximation for min span periodic planar RCP of asymptotic ratio α for periodic planar graphs. Note also that, the same algorithmic approach can be applied to obtain a radiocoloring assignment that approximates the *minimum order* within a ratio which tends to $\frac{5}{3}$ as well, as we show in (11).

Our results provide yet another natural $PSPACE$-complete problem whose optimization version is shown here to admit a polynomial time constant ratio approximation. This answers partially an interesting open question in Condon et al (6).

1.4 Previous Work

Several practical variations of FAP have been studied in the past for some useful families of graphs, e.g. (12; 8; 19; 3). Both versions of RCP, minimizing the span or the order, have proved to be NP-complete, even for the case of planar graphs, see (3; 9) and (23), respectively.

As it concerns the approximability of the problem, for min order RCP, in (9) a 2-approximation algorithm was presented for the case of planar graphs. This result was next improved in (2), providing a $\frac{9}{5}$-approximation for planar graphs of $\Delta(G) \geq 749$. These results were further improved in (19), providing a $\frac{5}{3}$-approximation algorithm for the problem. For min span RCP on planar graphs, the latest result is also applicable here, obtaining the same approximation.

Actually, in (19), the authors provide upper bounds (which imply polynomial time approximation algorithms) for a more general problem called $\lambda_{p,q}$-labeling. In the $\lambda_{p,q}$-labeling problem, we seek to find an assignment of integers to the vertices of the graph so that any vertices of distance 2 get integers that differ by at least p and any two vertices of distance 1 get integers that differ by at least q. The objective of the assignment is to minimize the span. Note that $\lambda_{p,q}$-labeling is equivalent to min span RCP when $p = 2$ and $q = 1$. Variations of $\lambda_{p,q}$-labeling for $p = 2, 1$ and $p = 0, 1, 2$ have been considered in (3) and more recently in (1), providing approximations for some interesting families of graphs, such as outerplanar graphs, graphs of bounded treewidth, permutation and split graphs.

A model for periodic graphs (called *l-dimensional periodic graphs*) was first presented by Orlin in (21). The model of periodic graphs considered in this work is similar to that of Orlin for the 1-dimensional case, $l = 1$ (also called *1-dimensional periodically specified graphs* or simply *periodically specified graphs*), when restricted to planar instances.

The complexity of various basic problems of periodically specified graphs was studied by Orlin (21) and Wanke (26). In (21; 18; 25) it is proved that the problems of Maximum Independent Set (MIS), Hamiltonial Path, Partition into Triangles, SAT, 3-coloring for periodically specified graphs are $PSPACE$-complete. The appoximability of basic problems on infinite periodic graphs was studied by several researchers ((5; 13; 22)) giving efficient algorithms for solving problems such as determining strongly connected components, testing the existence of cycles, bipartiteness, planarity and minimum cost spanning forests for periodically specified graphs.

Marathe et al, in (17), presented several $PSPACE$-hardness results and also efficient approximation schemes for general classes of both hierarchically and periodically specified problems. However, we remark that the general framework of Marathe et al (17) does not cover the $PSPACE$-completeness of min span RCP of periodic planar graphs. This is so because the methodology of the periodic 3-SAT variation used in (17) does not trivially transfer to the 3-coloring of periodic planar graphs which we use in our reduction (see their Theorem 6.5). Also, their approximation technique for periodically specified graphs (illustrated for the Maximum Independent Set problem) takes the union of partial solutions-subsets of the infinite graph and thus it does not consider all the vertices, which is not allowed in coloring problems.

1.5 Organization of the paper

We provide here an overview of the rest of this paper. In Section 2 we provide a useful Lemma about the structure of a linear periodic planar graph. In section 3 we give the $PSPACE$-completeness proof of min span radiocoloring for periodic planar graphs. In Section 4we present an efficient approximation algorithm for min span periodic planar radiocoloring. Finally, we discuss open problems and possible further research.

2 Embeddings of Periodic Planar Graphs

In this paper, we use the notion of an *embedding* of a planar graph.

Definition 8 Planar Embedding (of a periodic graph G**)***((20)): For each node v of G, there is an adjacency list, such that all neighbours of v appear in clockwise order with respect to an actual drawing of G.*

The following Lemma reveals important information about the structure of a linear periodic planar graph.

Definition 9 *For a linear periodic planar graph G, given by the pair $(\widetilde{G}, E_+)$, the graph* **Extended** $\widetilde{G}$ *is obtained by an iteration i of G, G_i (which is isomorphic to $\widetilde{G}$) and the set of edges connecting G_i with the previous and next iterations, sets E_{i-}, E_{i+} (each of which is equal to E_+).*

Lemma 10 *Any linear periodic planar graph G can be embedded in the plane by interchanging at most two different planar embeddings of Extended $\widetilde{G}$.*

PROOF. Observe first that there are cases where we need to interchange two different embeddings of the graph Extended $\widetilde{G}$ in order to draw a linear periodic planar graph preserving planarity. As an example, consider the periodic graph whose graph $\widetilde{G}$ in its finite specification is a single edge connecting two vertices a, b, and $E_+ = \{(a,b),(b,a)\}$, illustrated in Figure 2. We will show

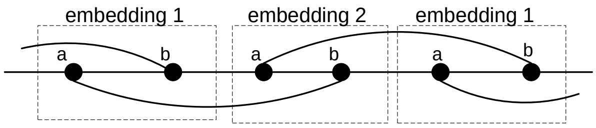

Fig. 2. An example of a graph which needs two different embeddings of Extended $\widetilde{G}$ in order to be drawn in the plane without edge crossings.

that interchanging at most two different embeddings of the graph Extended $\widetilde{G}$ is enough to draw any linear periodic planar graph G preserving planarity. In order to check whether it is possible that three embeddings to be required, we need to consider any 3 consecutive iterations of G. Recall that different embeddings may be introduced because of the connections between nodes of consecutive iterations. Hence, only nodes of the exterior face of each iteration will be involved. So, we can view the three consecutive iterations as three cycles. Moreover, we can consider three simple lines, since less edges are involved in the embeddings of consecutive iterations in the case of lines compared to cycles. Finally, observe that, in order for three or more planar embeddings to be needed, we have to consider at least 3 nodes of G_i, assume a, b, c. So, we can consider a line consisting of nodes a, b, c, having edges ab, bc (denoted by L_3). We distinguish 6 different 'drawings' of a line L_3 in the plane (see Figure 3).

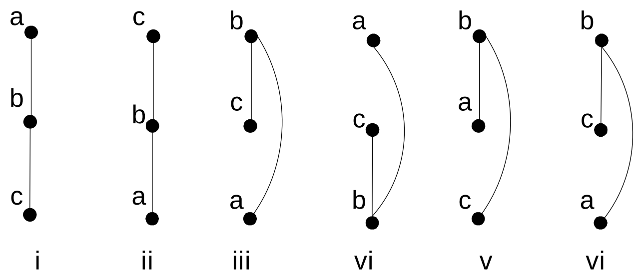

Fig. 3. The six possible embeddings of Line L_3.

Note that some of them are equivalent embeddings with respect to definition 8 of a planar embedding. However, we consider all 6 of them since for an iteration i each such drawing combined with sets of edges E_{i-} and E_{i+} can result in a distinct embedding of the graph Extended $\widetilde{G}$. In our case the Extended $\widetilde{G}$ is the graph obtained by L_3 of iteration i and a set of edges connecting L_3 with the previous and next iterations, (sets E_{i-}, E_{i+}). In the following, we use the term 'drawing' when we refer to any of these 6 drawings and the term 'embedding' when we refer to any of them together with an instance of sets E_{i-} and E_{i+}.

To check whether three embeddings may be needed, we need to check all possible triples of these drawings. For each such triple, we check all possible sets E_+ that can lead to a linear periodic graph. For each such set, we show that interchanging at most two of these six drawings in any two consecutive iterations, is enough to draw the infinite graph in the plane. Recall that two drawings (of consecutive iterations i and $i+1$) combined with the sets of edges connecting each iteration with next and previous iterations (sets E_{i-}, E_{i+} and $E_{(i+1)-}$, $E_{(i+1)+}$) result in two different embeddings of the graph Extended $\widetilde{G}$. Henceforth, interchanging two different embeddings of the graph Extended $\widetilde{G}$ we can draw the infinite periodic planar graph in the plane without crossings.

Assume iterations G_i, G_{i+1}, having nodes a_i, b_i, c_i, and $a_{i+1}, b_{i+1}, c_{i+1}$, respectively. Observe that, in order for three embeddings to be required (a) set E_+ should contain at least 3 edges between any two embeddings and (b) at least one node of each pair of nodes a_i, a_{i+1} , b_i, b_{i+1} or c_i, c_{i+1}, should have degree at least 1 in E_+. By exhaustive check of all possible cases, we conclude that in all cases, two embeddings are enough to draw the linear periodic graph in the plane, without edge crossings.

As one case of all possible cases (the other cases are similar), we consider the triple of drawings (i), (ii), (iii). For this triple, we check all possible sets of E_+ that lead to planarity. For each such set E_+, we show that at most two different embeddings of the graph Extended $\widetilde{G}$ are enough to draw the infinite graph in the plane. Observe first that there must not be an edge connecting c_i to any node of G_{i+1}, since this would eliminate the possibility of a third embedding. Also, there must be an edge from some node of G_i to c_{i+1}, since otherwise, two embeddings of the graph Extended $\widetilde{G}$ would be sufficient. Thus, there are three possible cases for the set of edges between any two consecutive iterations that preserve planarity.

A. There exists an edge (a_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), it is not possible to have both edges (a_i, a_{i+1}) and (b_i, b_{i+1}). Hence, the only cases remaining are to have edges $\{(a_i, b_{i+1}), (b_i, b_{i+1}), (b_i, a_{i+1})\}$ or edges $\{(a_i, b_{i+1}), (a_i, a_{i+1}), (b_i, a_i + 1)\}$. Both of them lead to a linear periodic planar graph by interchanging drawings (*i*) and (*ii*). Obviously, for any other subset of those cases at most two embeddings of the graph Extended $\widetilde{G}$ are sufficient to preserve the planarity of the periodic planar graph.

B. There exists an edge (b_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), it is not possible to have both edges (a_i, b_{i+1}) and (b_i, a_{i+1}). Hence, the only cases remaining are to have edges $\{(a_i, a_{i+1}), (a_i, b_{i+1}), (b_i, b_{i+1})\}$ or edges $\{(a_i, a_{i+1}), (b_i, a_{i+1}), (b_i, b_{i+1})\}$. Both of them lead to a linear periodic planar graph by using drawing (*ii*). Obviously, for any other subset of those cases at most two embeddings of the graph Extended $\widetilde{G}$ are sufficient to preserve the planarity of the periodic planar graph.

C. There exist both edges (a_i, c_{i+1}) and (b_i, c_{i+1}). In this case, by planarity (the resulting graph should be planar), the only possible extra edges are the following sets:
(-) $\{(a_i, b_{i+1}), (a_i, a_{i+1})\}$. Then we can use drawing (i).
(-) $\{(b_i, b_i+1), (b_i, a_{i+1})\}$. Then we can interchange drawings (i) and (ii).
(-) $\{(a_i, a_{i+1}), (b_i, b_{i+1})\}$. Then we can use drawings (iii).

By planarity, any other supersets of those cases are not possible. For any other subset of those cases at most two embeddings of the graph Extended $\widetilde{G}$ are sufficient. For the rest possible triples of drawings we get similar results. See (10) for verification of the Lemma through a computer program that checks all possible cases.

$\diamond$

Lemma 11 *Assume that we construct an infinite graph as in Definition 6 except that instead of* **(a3)** *we have that for any three consecutive points of the line, replacing each point by graph* $\widetilde{G}$ *and the edges among consecutive points by* E_+*, the resulting graph is planar. Then, the infinite graph thus constructed is a linear periodic planar graph.*

PROOF. The above requirement implies **(a3)** of Definition 6 because of Lemma 10. To see why, observe that there are two possible ways to draw a linear periodic planar graph: using the infinite sequence $\ldots, A, B, A, \ldots$ or $\ldots, B, A, B, \ldots$ (where A, B are the two planar embeddings of $\widetilde{G}$ needed). Our assumption guarantees that the coexistence of AB and BA graphs (together with E_+) has a planar embedding. Thus, by induction, any longer string will also have a planar embedding. $\diamond$

We study the following optimization version of the periodic planar radiocoloring problem:

Definition 12 min span RCP for a periodic planar graph G**:** *Given a periodic planar graph* $G = (\widetilde{G}(V, E_0), E_+)$*, find a radiocoloring assignment of minimum span for this graph. Denote this span by* $\lambda_{span}(G)$.

Note that optimization problems defined on such infinite graphs tend to be harder than the finite case (e.g. either $PSPACE$-hard or $NEXPTIME$-hard) as noticed by (6; 7).

3 The $PSPACE$-Completeness of min span RCP for Periodic Planar Graphs

We prove have that min span radiocoloring for periodic planar graphs is $PSPACE$-complete. In order to show this, we need to prove that a number of problems are $PSPACE$-complete. A 3-coloring of a periodic graph $G = (\widetilde{G}(V, E_0), E_+)$ is a function c that assigns a number from the set $\{1, 2, 3\}$ to each vertex of the graph G so that not two adjacent vertices get the same number. A 4-edge coloring of a periodic graph G is a function e that assigns a number from the set $\{1, 2, 3, 4\}$ to each edge of G so that no two adjacent edges get same numbers.

Let any iteration $G_i(V_i, E_i)$ of G and let E_{i+} the set of edges connecting any iteration G_i to the next iteration G_{i+1}. A *constant period 4-edge coloring* of a periodic graph G is a 4-edge coloring of G that assigns to each edge uv of E_i of any iteration G_i the same color as the color assigned to the corresponding edge $u'v'$ of any other iteration G_j of G. Also, such a 4-edge coloring assigns to each edge of E_{i+} the same color as the color of the corresponding edge of E_{j+} of any other iteration G_j. As a result, such a 4-edge coloring can be described using finite space (using the 4-edge coloring of G_i and E_{i+}).

Lemma 13 *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ is 3-colorable (also called periodic planar 3-coloring) is $PSPACE$-complete.*

PROOF. (a) *Membership in $PSPACE$:* The proof that this problem is in $PSPACE$ is similar to that for the PERIODIC SAT, (25). Suppose that the given periodic graph is 3-colorable, and consider a valid 3-coloring of it. This assignment consists of a two-way infinite $\ldots, T_i, T_{i+1}, T_{i+2}, \ldots$ of valid 3-coloring assignments to the various blocks of nodes (one for each iteration). Each T_i is an element of $\{1, 2, 3\}^n$, where $n = |V|$ is the number of vertices of the graph G_i.

The ith *chunk*, where i is any integer, is the pair (T_i, T_{i+1}), of two consecutive valid 3-coloring assignments. Since there are 3^{2n} possible different chunks, there must be two chunks, not further than 3^{2n} from each other, that are identical. That is $(T_i, T_{i+1}) = (T_j, T_{j+1})$ for some i and j between $i + 2$ and $i+3^{2n}$. But this means that there is a 3-coloring assignment consisting of a two-way infinite repetition of $(T_i, T_{i+1}, \ldots, T_{j-1})$. We conclude that *if a periodic graph G is 3-colorable, then it has a periodic 3-coloring assignment with period at most exponential in the number of nodes of one iteration.* Using this crucial observation, we can show that a polynomial-space machine can guess and check any 3-coloring of the graph: using non-determinism we can guess valid assignments $T_1, T_2, \ldots$, always remembering the last two. After we guess T_i we check that all nodes in iteration $i-1$ are still properly colored. Once we have successfully guessed $T_{3^{2n}+2}$ we accept: We know that there is a periodic 3-coloring assignment on G.

(b) *The $PSPACE$-completeness proof:* In order to show the completeness, we reduce from 3-coloring of periodic general graphs, which is known to be $PSPACE$-complete ((25)). We use the transformation used by (24) to prove the NP-completeness of PLANAR-3-COLORING reducing it from 3-COLORING. We describe it next:
Transformation: Consider any graph G. Construct a new *planar* graph G' as follows: replace any edge-crossing of G with the gadget of Figure 4, where H is a subgraph presented in the same Figure. The vertices of the subgraph are named as shown in Figure 4.

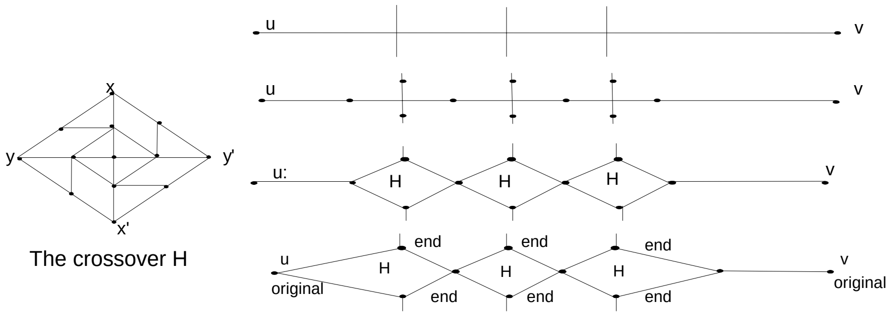

Fig. 4. The Crossover H and the construction of a planar G' from a given periodic graph G, using the crossover H

Let any periodic graph G that we are asked whether it is 3-colorable. Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G. Call the vertices of G_i (set V_i), *original vertices of* G_i. Call the edges of G_i (set E_i) as *inside edges of* G_i. Call the edges of set E_{i+} as *outside edges of* G_i. Similarly, if a vertex in V_i has no neighbour vertex in V_{i+1}, then the vertex is called *inside vertex of* G_i. Otherwise, that is, when the vertex has a neighbour in V_{i+1}, call it *outside vertex of* G_i.

From G, we construct a new graph G' according to the following Procedure:
1. Apply the Transformation on the graph G_i.
2. Apply the Transformation on the subgraph of G obtained the edges connecting G_i to G_{i+1} (set E_{i+}).

Consider now any outside edge of G_i. If there are no crossings on it, the edge remains the same in G'. Such edges belong to a set called $OutNoCrossEdge_{G_i}$. Otherwise, that is, when there are some crossings on the edge, the Procedure replaces it by a sequence of crossovers (one for each crossing), as shown in Figure 4. Call such gadgets, *outside sequences of crossovers of* G_i, denoted by

$Out\_SC_{G_i}$.

Consider any inside edge uv of G_i. If there are no crossings on it, the edge remains the same in G'. Such edges belong to a set called $InNoCrossEdge_{G_i}$. Otherwise, that is, when there are some crossings on the edge, the Procedure replaces it by a sequence of crossovers (one for each crossing), as shown in Figure 4. If none of its *end* vertices belongs to a crossover corresponding to an outside edge of G_{i-1}, then call such a gadget, *inside sequences of crossovers, considering iteration* G_i, denoted by $In\_SC_{G_i}$.

Let now any inside edge of G_{i+1}. If there are no crossings on it, the edge remains the same in G'. Otherwise, that is, when there are some crossings on the edge, the Procedure replaces it by a sequence of crossovers (one for each crossing), as shown in Figure 4. If at least one of its end vertices belongs to a crossover corresponding to an outside edge of G_i, then call such a gadget *inside sequence of crossovers considering iteration* G_{i+1}, denoted by $In\_SC_{G_{i+1}}$. Finally, for any sequence of crossovers SC, defined above, let $Orig_{SC}$ the original vertices of iteration G_{i+1} contained in this sequence of crossovers SC.

Now define a graph $G'_i(V'_i, E'_i)$ as follows:
$V'_i = V_i \cup \ (V(Out\_SC_{G_i}) - Orig_{Out\_SC_{G_i}}) \ \cup \ V(In\_SC_{G_i}) \cup (V(In\_SC_{G_{i+1}}) - Orig_{In\_SC_{G_{i+1}}})$,
$E'_i = InNoCrossEdge_{G_i} \cup E(Out\_SC_{G_i} - Orig_{Out\_SC_{G_i}})) \cup E(In\_SC_{G_i})$
$\cup E(In\_SC_{G_{i+1}} - Orig_{In\_SC_{G_{i+1}}})$.
Also define a set of pairs of nodes, called E'_{i+}, as follows:
$(v,u) \in E'_{i+} \ if \ \ vu \ \in Out\_SC_{G_i} \ \ and \ \ u \in V_{i+1}$,
$(v,u) \in E'_{i+} \ if \ \ vu \ \in In\_SC_{G_{i+1}} \ \ and \ \ u \in V_{i+1}$,
$(v,u) \in E'_{i+} \ if \ vu \ \in OutNoCrossEdge_{G_i} \ \ and \ \ u \in V_{i+1}$.

Applying the Procedure for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of pairs of nodes E'_{i+}. Thus, we can apply the Procedure for only one iteration of G. The resulting pair (G'_i, E'_{i+}) defines a *periodic graph* given by $G' = (G'_i(V'_i, E'_i), E'_{i+})$ and is obtained in polynomial time to the specification of the graph G. Observe also that the Procedure eliminates all edge-crossings so the new graph is also planar.

We next show that if the initial periodic graph G is 3-colorable, then the periodic *planar* G' is also 3-colorable. Our reduction replaces each edge-crossing of the periodic graph with the gadget of (24). In (24), it is shown that such a replacement guarantees than the new graph obtained is 3-colorable if and only if the original is 3-colorable. Thus, the statement holds for the new periodic graph G' obtained by our reduction. This completes the $PSPACE$-completeness proof. $\diamond$

Lemma 14 *The problem of deciding whether a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree four is 3-colorable is $PSPACE$-complete.*

PROOF. We prove that the problem is in $PSPACE$ using arguments similar to those used above to prove the membership of periodic planar 3-coloring in the class $PSPACE$.

We reduce planar 3-coloring of a periodic graph with a maximum degree 4 from periodic planar 3-coloring, which we proved that it is $PSPACE$-complete in Lemma 13. We use the transformation used to prove the NP-completeness of 3-COLORING of a planar graph G with a maximum degree 4 in (24). That reduction reduces the problem from the PLANAR-3-COLORING. Recall the gadget used in (24):
Gadget: Consider a vertex substitute, called H_3, shown in Figure 5. The graph H_3 has three "outlets" labeled 1, 2, 3. It is designed so that all of its outlets should take the same color. Moreover, each outlet has degree 2. A vertex of degree k in G is replaced by a subgraph H_k, which is defined by $k-2$ repetitions of the subgraph H_3, as shown in Figure 5, for the case where $k = 5$. In the subgraph H_k all of its outlets should be colored with the same color and each of them is of degree 2.

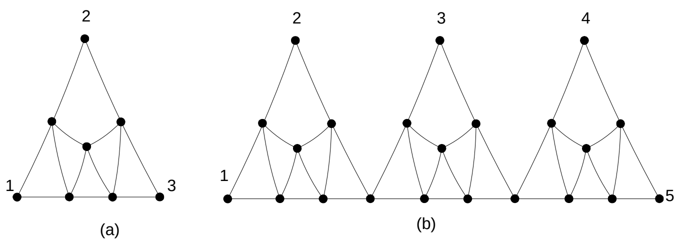

Fig. 5. The subgraphs H_3 (a) and H_5 (b)

Let any periodic planar graph G of arbitrary maximum degree that we are asked whether it is 3-colorable. We construct a new graph G' of maximum degree is 4 which we show that it is a periodic planar graph. Consider any iteration $G_i(V_i, E_i)$ of the periodic graph G. We replace the vertices of the graph with the Gadget above, similar to (24), accordinh to the following Procedure:
1. Replace any vertex v of G_i of degree k by the subgraph H_{k_v}.
2. Replace each edge (v, u_j), with $v, u_j \in V_i$, by an edge joining u_j to the j-th outlet of H_{k_v}.
3. Replace each edge (v, u_j), with $v \in V_i$ but $u_j \in V_{i+1}$, by an edge joining u_j to the j-th outlet of H_{k_v}.

Define a graph $G'_i(V'_i, E'_i)$ as follows:
$\forall u \in V_i : V(H_{k_u}) \subseteq V'_i$, $\forall u \in V_i : E(H_{k_u}) \subseteq E'_i$ and $\forall (u, v_j) \in E_i$ *and* $u, v_j \in$

$V_i \; : (v_j, j) \in E_{i'}$, where j is the j-th outlet of H_{k_u} and k is vertex's u degree. Also, define a set of pairs of nodes, called E'_{i+}, of as follows: $\forall (u, v_j) \in E_i$ *and* $u \in V_i$ *but* $v_j \in V_{i+1} : (v_j, j) \in E'_{i+}$.

Applying the Procedure for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of pairs of nodes E'_{i+}. Thus, we can apply the Procedure for only one iteration of G. The resulting pair (G'_i, E'_{i+}) defines a *periodic graph* given by $G' = (G'_i(V'_i, E'_i), E'_{i+})$ and is obtained in polynomial time to the specification of the graph G. Observe also that the graph obtained by the Procedure is a periodic planar graph of maximum degree 4.

We need to prove that if the initial periodic graph G is 3-colorable, then the periodic planar G' of maximum degree 4, is also 3-colorable. Our reduction replaces each vertex of the periodic graph with the same gadgets as in (24). In (24), it is shown that such a replacement guarantees than the new graph obtained is 3-colorable if and only if the original is 3-colorable. Thus, the statement holds for the new periodic graph G' obtained by our reduction. This completes the $PSPACE$-completeness proof. $\diamond$

Lemma 15 *The problem of 3-coloring a periodic planar graph with a given constant period 4-edge coloring is $PSPACE$-complete.*

PROOF. The membership in $PSPACE$ can be shown using arguments similar those used to prove the membership of 3-coloring of periodic graphs in the class $PSPACE$ of Lemma 13.

We use a transformation from the 3-coloring for planar periodic graphs with maximum degree 4 which was proved to be $PSPACE$-complete in Lemma 3, similar to (24) used to show NP-completeness of 3-COLORING of planar graphs with maximum degree 4. Consider any iteration $G_i(V_i, E_i)$ of the peri-

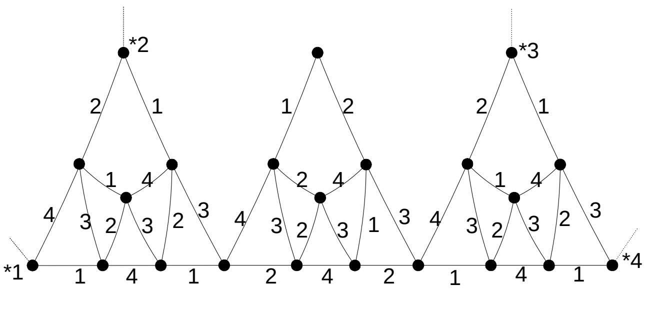

Fig. 6. The subgraph S

odic graph G. From G, we construct a periodic planar graph G' according to the following Procedure:

1. Replace every vertex u of G_i by a copy of the subgraph S, used in (24),

shown in Figure 6.
2. Consider any vertex u in V_i which is replaced in G' by a subgraph S. We replace each edge (u, v), with $v \in V_i$, incident to u with an edge joining v with one of the four marked vertices (with an $*i$, for $i = 1 \ldots 4$) in the subgraph S such that the graph stays planar.
3. Similarly, we replace each edge (u, v), with $u \in V_i$ but $v \in V_{i+1}$, incident to u with an edge joining v with one of the four marked vertices in the subgraph S (replacing u in G') so that the graph stays planar.

Define now a graph $G'_i(V'_i, E'_i)$ as follows:
$\forall u \in V_i : V(S_u) \subseteq V'_i,\ \forall\ v\ \in V_i\ :\ E(S_u) \subseteq E'_i$ and $\forall (u, v_j) \in E_i$ *and* $u, v_j \in V_i : (v_j, j) \in E_{i'}$, where j is the j-vertex marked as $*j$ in S_u.
Also define a set of pairs of nodes, called E'_{i+}, of as follows: $\forall (u, v_j) \in E_i$ *and* $u \in V_i$ *but* $v_j\ \in V_{i+1} : (v_j, j) \in E'_{i+}$.

Applying the Procedure for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of pairs of nodes E'_{i+}. Thus, we can apply the Procedure for only one iteration of G. The resulting pair (G'_i, E'_{i+}) defines a *periodic graph* given by $G' = (G'_i(V'_i, E'_i), E'_{i+})$ and is obtained in polynomial time to the specification of the graph G.

Let G' be the resulting periodic graph. An 4-edge coloring of G' can be constructed by coloring the edges in the replacement subgraph S as in Figure 6, and coloring the original edges of G (the dotted lines in Figure 6) by the following procedure: if e is an edge between two subgraphs, since both endpoints have degree three and one edge is already colored with 1, there is at least one color from { 2, 3, 4 } left for the edge.

We next prove that the 4-edge coloring of the periodic graph G' can be constructed in time polynomial to the size of one iteration of G'. That is, to prove that it is a constant period 4-edge coloring. To see why, consider any iteration $G'_i(V'_i, E'_i)$ of G'. The edge coloring procedure described above may apply to produce a 4-edge coloring of edges of set E'_i in time polynomial to $n = |V'_i|$. Now use the same edge-coloring assignment of E'_i to edge-color the edge sets E'_j of each other iteration G'_j of the periodic graph G'. Observe, that this assignment results to no conflicts between successive iterations because none of the edges of E'_j is adjacent to an edge of $E'_{(j+1)}$. Consider next, the edges of set E_{i+} connecting some vertices of G_i to some vertices of G_{i+1}. The procedure described above may apply to produce a 4-edge coloring of edges of set E_{i+} in time polynomial to $n = |V'_i|$. Now as before, use the same edge-coloring assignment of E'_{i+} to edge-color the edges of sets E'_{j+} of the each other iteration G'_j of the periodic graph G'. Observe again that, this assignment results to no conflicts between successive iterations because none of the edges of E'_{j+} is adjacent to an edge of $E'_{(j+1)+}$. *Hence this is a constant period 4-edge coloring G' and thus it can be computed in time polynomial to the size the specification* *of G'.*

Now, we prove that if the initial periodic planar graph G, with a given 4-edge coloring is 3-colorable, then the periodic planar G', is also 3-colorable and has 4-edge coloring. Our reduction replaces each vertex of the periodic graph with the same gadgets as in (24). In (24), it is shown that such a replacement guarantees than the new graph obtained is 3-colorable if and only if the original is 3-colorable. Thus, the statement holds for the new periodic graph G' obtained by our reduction. This completes the $PSPACE$-completeness proof. $\diamond$

Next, we prove the main Theorem using the above results. We prove that the min span radiocoloring for periodic planar graphs is $PSPACE$-complete, by transforming it from 3-coloring of periodic planar graphs.

Theorem 16 *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$, of maximum degree seven, whose graph $\widetilde{G}$ in its finite specification $(\widetilde{G}, E_+)$ is a planar bipartite graph, can be radiocolored using a span of at most 9, is $PSPACE$-complete.*

PROOF. (a) *Membership in $PSPACE$:* The proof that it is in $PSPACE$ is the same as for PERIODIC SAT, (25). Suppose that the given periodic graph which can be radiocolored using a span of size k and consider a valid radiocoloring assignment of it. This assignment consists of a two-way infinite $\ldots, T_i, T_{i+1}, T_{i+2}, , \ldots$. of valid radiocoloring assignments to the various blocks of nodes (one for each iteration). Each T_i is an element of $\{1, 2, \ldots, k\}^n$, where $n = |V|$ is the number of vertices of the graph G_i.

The ith iteration, where i is any integer, is the pair (T_i, T_{i+1}), of two consecutive valid radiocoloring assignments. Since there are k^{2n} possible different chunks, there must be two chunks, not further than k^{2n} from each other, that are identical. That is $(T_i, T_{i+1}) = (T_j, T_{j+1})$ for some i and some j between $i+2$ and $i + k^{2n}$. But this means that there is a a valid radiocoloring assignment consisting of a two-way infinite repetition of $(T_i, T_{i+1}, \ldots, , T_{j-1})$. We conclude that *if a periodic graph G can be radiocolored wit a span of at most 8, then it has a periodic radiocoloring assignment with period at most exponential in the number of nodes of one iteration.* This crucial observation, we can show that a polynomial-space machine can guess and check any 3-coloring of the graph: using Using non-determinism we can guess valid assignments $T_1, T_2, \ldots,$ always remembering the last two. After we guess T_i we check that all nodes in the $i - 1$ iteration are still valid radiocolored. Once we have successfully guesses $T_{k^{2n+2}}$ we accept: We know that there is a periodic radiocoloring assignment that uses a span of size no more than k.

(b) *The $PSPACE$-completeness proof:* We utilize the transformation used in

(3) to prove the NP-completeness min span radiocoloring of ordinary planar graph. That result reduces from 3-COLORING of planar graphs with a given 4-edge coloring. Here we reduce from 3-coloring of periodic planar graphs with a given 4-edge coloring, which we proved that it is $PSPACE$-complete in Lemma 15.

Suppose we are given a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ with a 4-edge coloring f. Consider the subgraph of G obtained by any iteration $G_i(V_i, E_i)$ of the periodic graph G and the edges E_{i+} connecting it to the next iteration. For this subgraph, we apply the transformation of (3), which we also describe here.

1. For each vertex in V_i of G_i, we add a new vertex in G', called *original* vertex.

Next we subdivide every edge kl of E_i or E_{i+} to get vertex u_{kl}. The resulting vertices are named as follows:

2. Every vertex resulting from the subdivision of an edge colored 1 is called *0-vertex.* **3.** Every vertex resulting from the subdivision of an edge colored 2 is called *1-vertex.* **4.** Every vertex resulting from the subdivision of an edge colored 3 is called *7-vertex.* **5.** Every vertex resulting from the subdivision of an edge colored 4 is called *8-vertex.*

6. Now, if an original vertex has degree less than four, it receives new neighbors: if original vertex v has no neighbor that is a 0-vertex, we add one new vertex and connect it only to v. Call this vertex an *extra 0-vertex.* Similarly, original vertices that have no neighbor that is a 1-vertex, 7-vertex, or 8-vertex get a new neighbor of degree one that is an *extra 1-vertex, extra 7-vertex*, or *extra 8-vertex.*

7. Now we add, to every 0-vertex and 8-vertex, five new neighbors of degree one each, as in Figure 7.

8. To every extra 0-vertex and extra 8-vertex, we add six new neighbors of degree one each, as in Figure 7.

9. To every 1-vertex and 7-vertex, we add two subtrees of a form, as shown in 8. To every extra 1-vertex and extra 7-vertex we add three of these subtrees, as also shown in Figure 8.

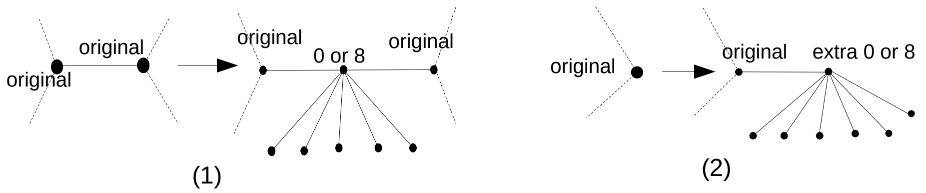

Fig. 7. The vertices added to each 0 or 8-vertex (rule 1), and the vertices added to each extra 0 or 8-vertex (rule 2).

Call the induced subgraph defined by the original vertex k, the extra vertices added for it and their neighbors, as O_k. Also, call the induced subgraph,

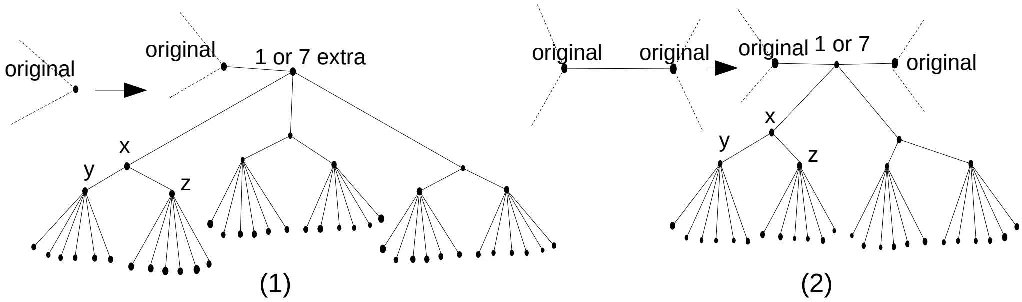

Fig. 8. The vertices added to each 1-vertex and 7-vertex (rule 1), to each extra 1-vertex and extra 7-vertex (rule 2)

defined by a new vertex u_{kl} (obtained by the edge subdivision of edge kl) and the new vertices added to this vertex ((by rules (7),(9)), as subgraph S_{kl}.

Using rules 1-9, we define a graph $G'_i(V'_i, E'_i)$ as follows:

(a) $\forall k \in V_i : V(O_k) \subseteq V'_i$ (by rules 1, 8, 9),
(b) $\forall (k,l) \in E_i$ *and* $k,l \in V_i : u_{kl} \in V'_i$ (by rules 2, 3, 4, 5),
(c) $\forall k \in V_i : E(O_k) \subseteq E'_i$ (by rules 8, 9),
(d) $\forall (k,l) \in E_i : V(S_{kl}) \subseteq V'_i$ (by rules 7, 9),
(e) $\forall (k,l) \in E_i : E(S_{kl}) \subseteq E'_i$ (by rules 7, 9),
(f) $\forall (k,l) \in E_i$ *and* $k,l \in V_i : (k, u_{kl}), (u_{kl}, l) \in E'_i$ (by rules 2, 3, 4, 5),
(g) $\forall (k,l) \in E_i$ *and* $k \in V_i$ *but* $l \in V_{i+1} : (k, u_{kl}) \in E'_i$ (by rules 2, 3, 4, 5).

Also, define a set of pairs of nodes, called E'_{i+}, as follows:
$(k,l) \in E_i$ *and* $k \in V_i$ *but* $l \in V_{i+1} : (u_{kl}, l) \in E'_{i+}$ (by rules 2, 3, 4, 5).

Applying the Procedure, for any other iteration G_j of G and the edges connecting G_j to G_{j+1}, we get the same graph $G'_i(V'_i, E'_i)$ and set of pairs of nodes E'_{i+}. Thus, we can apply the Procedure for only one iteration of G. The resulting pair (G'_i, E'_{i+}) defines a *periodic graph* given by $G' = (G'_i(V'_i, E'_i), E'_{i+})$ and is obtained in polynomial time to the specification of the graph G. Observe also that, if the initial graph G is a planar graph, then the resulting graph G' is also a planar graph. To see why, observe that the extra edges added to G' form a star or a tree. Hence the planarity of the initial graph G remains.

We next show that G is 3-colorable, if and only if $\lambda_{span}(G') \leq 9$. This is true because our reduction replaces each vertex of the periodic graph with the same gadgets as in (3). In (3), it is shown that such a replacement guarantees than the new graph obtained can be radiocolored using a span of size at most 9 if and only if and only if the original is 3-colorable. Thus, the statement holds for the new periodic graph G' obtained by our reduction. This completes the $PSPACE$-completeness proof. $\diamond$

As in (3), it is possible to generalize the result as follows:

Theorem 17 *Let $r \geq 8$ be an even integer. The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree $r - 2$ can be radiocolored using a span of size at most r is $PSPACE$-complete.*

4 An Efficient, Constant Ratio Approximation Algorithm for min span RCP for Periodic Planar Graphs

We present an efficient time, constant ratio approximation algorithm that approximates the min span radiocoloring problem for periodic planar graphs with the same ratio as the ratio obtained by the best known approximation algorithm for planar graphs (which we use as a subroutine for the finite specification), for the same problem.

We first present some useful notation. Recall that the minimum span of a radiocoloring of a graph G is denoted as $\lambda_{span}(G)$. The span used by a (not necessarily optimal) radiocoloring algorithm of G is denoted as $\lambda'_{span}(G)$. Let I an instance of an optimization problem P when a measurement O of the solution is of concern. The *approximation ratio* $r_A(I)$ of an algorithm A for the instance I of the problem is the ratio of O computed by Algorithm A, denoted as $O_A(I)$, over the optimal O, denoted as $O^*(I)$, i.e. $r_A(I) = \frac{O_A(I)}{O^*(I)}$.

4.1 The modified graph

Consider the following partition of the periodic graph G: group together every four consecutive iterations of the graph, call the $j - th$ such group $G_{group\, j}$. More specifically,

$$G_{group\, j} = \{G(i) \bigcup G(i+1) \bigcup G(i+2) \bigcup G(i+3)\}, \forall i = \ldots, -7, -3, 1, 5, 9, \ldots$$

where $j = 1, 2, 3, 4, \ldots$, (respectively, i.e. $j = \lceil i/4 \rceil$).

See Figure 9 for an example of the partition. Consider the graph $G_{group\, j}$ of G. Denote the first graph of the group as $G_{(j)1}$ or G_1, the second as $G_{(j)2}$ or G_2 and so on until the fourth. The algorithm uses the following graph.

Definition 18 *The graph $G'_{group\, j}$ has the same vertex and edge set as the graph $G_{group\, j}$ except from the following modifications on the first and the fourth graphs of group $G_{group\, j}$: Consider an edge $uv \in E_{4+}$ of graph G_4.*

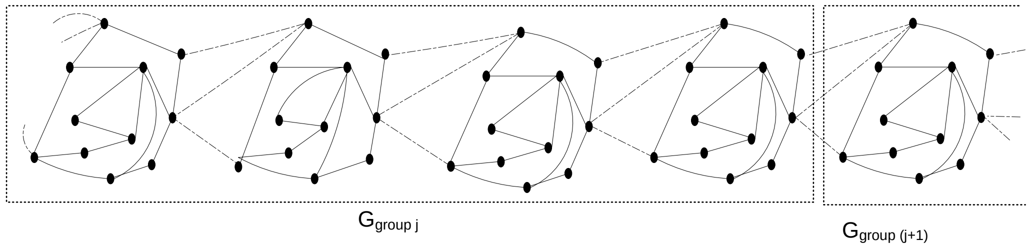

Fig. 9. The partitioning of graph G into groups of four consecutive iterations in each group.

Recall that $u \in V_4$ and the vertex v belongs to the next iteration of G, that is $v \in V_{(j*4)+1}$. For each such edge uv of the graph G_4 we do the following:

- *Delete edge uv from G_4.*
- *Add a new edge uv' connecting the vertex $u \in G_4$ to vertex v', where $v' \in V_1$ is the corresponding to v ($v \in V_{(j*4)+1}$) vertex in graph G_1. Recall that the graphs $G_1, V_{(j*4)+1}$ are isomorphic.*
- *Delete edge $u''v'$ of graph G_1, where $v' \in G_1$ and u'' is the corresponding to u vertex ($u \in V_4$) in iteration G_{j*4-1} of G.*

The produced graph for the example of a periodic graph of Figure 9 is illustrated in Figure 10. The graph $G'_{group\,j}$ has two critical properties compared to the initial periodic planar graph G: (i) it has the same maximum degree as the initial graph G, i.e. $\Delta(G'_{group\,j}) = \Delta(G)$ and (ii) as the next Lemma proves, it is a planar graph.

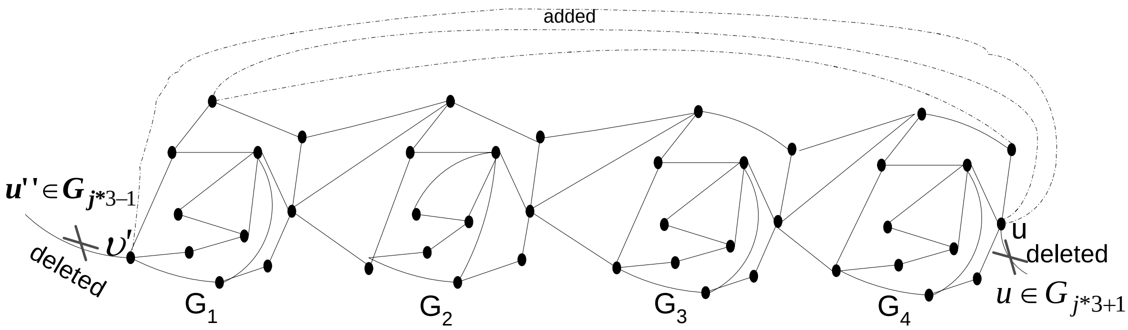

Fig. 10. The graph $G'_{group\,j}$ produced by the *Group j* of the periodic graph G.

Lemma 19 *The modified graph $G'_{group\,j}$ is a planar graph.*

PROOF. Observe that, by Lemma 10 any four consecutive iterations can be drawn in a plane using embeddings E_1, E_2, E_1, E_2, where it might be that $E_1 = E_2$. The modified graph $G'_{group\,j}$ is the same as graph $G_{group\,j}$ (which is planar), with the only difference that G_4 is connected to G_1 instead of G_{j*4+1}. Assuming that embedding E_1 is used be G_1, graph G_4 uses embedding E_2. In the periodic graph G, G_4 is connected to G_{j*4+1} which uses embedding E_1. In the modified graph $G'_{group\,j}$, G_4 is connected to G_1, which also uses embedding E_1. Thus, there is no crossing in edges connecting G_4 to G_1. We conclude that $G'_{group\,j}$ is a planar graph. $\diamond$

4.2 The Periodic Radiocoloring Partitioning Algorithm (PRPA)

The following definition is also utilized by the Algorithm. The definition uses the observation that $\lambda_{span}(G) \geq \Delta(G)$.

Definition 20 RC Algorithm: *Let an RC Algorithm be any known min span radiocoloring polynomial time approximation algorithm for finite planar graphs with performance ratio R (when $\Delta(G)$ is used as a lower bound). That is, there are constants $R > 1$ and b such that,*

$$\Delta(G) \leq \lambda_{span}(G) \leq \lambda'_{span}(G) \leq R \cdot \Delta(G) + b$$

For example, the algorithm of (19) is an RC algorithm with $R = \frac{5}{3}$ and $b = 90$.

The proposed radiocoloring algorithm for a periodic graph G, called Periodic Radiocoloring Partitioning Algorithm (PRPA), can now be described.

Algorithm PRPA:

(1) `Run an` RC `algorithm, on graph` $G'_{group\,j}$.
`Let` $\lambda'_{span}(G'_{group\,j})$ `be the span obtained by` RC `on` $G'_{group\,j}$.

(2) `For all` $j = 1, 2, \ldots$ `color the four graphs` $G_{(j-1)*4+1}$, $G_{(j-1)*4+2}$, $G_{(j-1)*4+3}$, , $G_{(j-1)*4+4}$ `of the group` $G_{group\,j}$ `as follows:`
`Set the color of each vertex of graph` $G_{(j-1)*4+k}$, $k = 1, 2, 3, 4$ `to the color of its corresponding vertex, in` V_k `of` $V(G'_{group\,j})$.

Note that Step 2 produces a radiocoloring of the whole periodic graph G with span $\lambda'_{span}(G'_{group\,j})$. Thus, the algorithm computes $\lambda'_{span}(G) = \lambda'_{span}(G'_{group\,j})$.

4.3 Correctness and Performance of Algorithm

Theorem 21 (**Correctness**) *The algorithm $PRPA$ produces a radiocoloring of a periodic graph G.*

PROOF. We prove that there is no conflict either between the colors of vertices inside any group $G_{group\,j}$ of G or between the colors of vertices in neighbour groups of G.

We first prove that there is no conflict between the colors of vertices inside a group $G_{group\,j}$. Any two vertices of the same group get the same colors as the colors of their corresponding vertices in $G'_{group\,j}$. Also, their distance in $G_{group\,j}$ is the same distance the distance of their corresponding vertices in $G'_{group\,j}$, by the construction of the latter graph. Since the radiocoloring assignment RC, computed for $G'_{group\,j}$ is correct, there is no conflict between the colors of the two vertices.

Next, we check the colors of vertices in neighbour groups. Consider any vertex u of $G_{group\,j}$ and a vertex v of group $G_{group\,j+1}$. Consider also the corresponding to those two vertices in $G'_{group\,j}$, u' and v', respectively. Recall that the distance between u' and v' is the same distance as that of u' and v', by the construction of $G'_{group\,j}$. Recall also that u and v get the same colors as the colors of their corresponding vertices u' and v'. Since the radiocoloring assignment RC, computed for $G'_{group\,j}$ is correct, there is no conflict between the colors of vertices u and v.

$\diamond$

Theorem 22 (Performance) *The Algorithm $PRPA$ runs in time $O(T(RC))$ and approximates the optimal span of any periodic planar graph G within an asymptotic ratio of R, where R is the approximation ratio obtained by the algorithm RC for finite planar graphs and $T(RC))$ is the time needed for the RC Algorithm to run on $G'_{group\,j}$.*

PROOF. Recall that the graph $G'_{group\,j}$ has maximum degree $\Delta(G'_{group\,j}) = \Delta(G)$.

Note that, $\lambda'_{span}(G) = \lambda'_{span}(G'_{group\,j}) \leq R \cdot \Delta(G'_{group\,j}) + b$, by the definition of the RC Algorithm.

Since, $\Delta(G'_{group\,j}) = \Delta(G)$, we get $\lambda_{span}(G) \leq \lambda'_{span}(G) \leq R \cdot \Delta(G) + b$.

Also, since $\lambda_{span}(G) \geq \Delta(G)$, we get that, $1 \leq \frac{\lambda'_{span}(G)}{\lambda_{span}(G)} = r_{PRPA}(G) \leq \frac{\lambda'_{span}(G)}{\Delta(G)} \leq R + \frac{b}{\Delta(G)}$.

Finally, since $\Delta(G) \geq \Delta(\widetilde{G})$, we have $1 \leq r_{PRPA}(G) \leq R + \frac{b}{\Delta(\widetilde{G})}$.

Also, the algorithm, clearly, needs $O(T(RC))$ time, where $T(RC)$ is the time needed for algorithm RC to run on $G'_{group\,j}$. $\diamond$

Note 2 *If the RC Algorithm is the algorithm in (19), then algorithm $PRPA$ has $R = \frac{5}{3}$ and $b = 90$ and runs in $O(n(\Delta(G_i) + |E_+|))$ time, where $n = |V_i|$.*

5 Extensions and Future Work

In (11), we proved that min order radiocoloring for periodic planar graphs is also $PSPACE$-complete using by similar reductions. In the same work we provide a 5/3-approximation algorithm for the problem. That algorithm is the same as $PRPA$ algorithm with the only difference that in Step 1, we apply a known min order radiocoloring algorithm (instead of a RC algorithm) of approximation ratio R (when $\Delta(G)$ is used as a lower bound). When we utilize the algorithm of (19) as the min order radiocoloring algorithm, the same analysis gives that the modified $PRPA$ algorithm approximates the min order RCP of G within an asymptotic approximation ratio of $\frac{5}{3}$.

An important open problem either the existence of a polynomial approximation scheme for min span RCP and/or min order RCP for periodic planar graphs or a proof that such a scheme does not exist.

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