On Radiocoloring Hierarchically Specified Planar Graphs: $\mathcal{PSPACE}$-completeness and Approximations

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Abstract. Hierarchical specifications of graphs have been widely used in many important applications, such as VLSI design, parallel programming and software engineering. A well known hierarchical specification model, considered in this work, is that of Lengauer [9, 10], referred to as *L-specifications.* In this paper we discuss a restriction on the L-specifications resulting to graphs which we call Well-Separated (WS). This class is characterized by a polynomial time (to the size of the specification of the graph) testable combinatorial property.

In this work we study the Radiocoloring Problem (RCP) on WS L-specified hierarchical planar graphs. The optimization version of RCP studied here, consists in assigning colors to the vertices of a graph, such that any two vertices of distance at most two get different colors. The objective here is to minimize the number of colors used. This problem is equivalent to the problem of vertex coloring the square of a graph G, G^2, where G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2.

We first show that RCP is $\mathcal{PSPACE}$-complete for WS L-specified hierarchical planar graphs. Second, we present a polynomial time 3-approximation algorithm as well as a more efficient 4-approximation algorithm for RCP on graphs of this class.

We note that, the best currently known approximation ratio for the RCP on ordinary (non-hierarchical) planar graphs of general degree is 2 ([6, 1]). Note also that the only known results on any kind of coloring problems have been shown for another special kind of hierarchical graphs (unit disk graphs) achieving a 6-approximation solution [13].

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1 Introduction, Our Results and Related Work

1.1 Motivation

Many practical applications of graph theory and combinatorial optimization in CAD systems, VLSI design, parallel programming and software engineering involve the processing of large (but regular) objects constructed in a systematic manner from smaller and more manageable components. As a result, the graphs that abstract such circuits (designs) also have a regular structure and are defined in a systematic manner using smaller graphs. The methods for specifying such large but regular objects by small specifications are referred to as *succinct specifications*. One way to succinctly represent objects is to specify the graph hierarchically. Hierarchical specifications are more concise in describing objects than ordinary graph representations. A well known hierarchical specification model, considered in this work, is that of Lengauer, introduced in [9, 10], referred to as *L-specifications*.

In modern networks, Frequency Assignment Problems (FAP) have important applications in efficient bandwidth utilization, by trying to minimize the number (or the range) of frequencies used, in a way that however keeps the interference of nearby transmitters at an acceptable level. Problems of assigning frequencies in networks are usually abstracted by variations of coloring graphs. An important version of Frequency Assignment Problems is the Radiocoloring Problem (RCP). The optimization version of RCP studied here, consists in assigning colors (frequencies) to the vertices (transmitters) of a graph (network), so that any two vertices of distance at most two get different colors. The objective here is to minimize the number of distinct colors used.

In this work we study RCP on L-specified hierarchical graphs. Note that RCP is equivalent to the problem of vertex coloring the square of a graph G, G^2, where G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2. We study here planar hierarchical graphs.

Also, our interest in coloring the square of a hierarchical planar graph is inspired by real communication networks, especially wireless and large ones, that may be structured in a hierarchical way and are usually planar.

1.2 Summary of Our Results

We investigate the computational complexity and provide efficient approximation algorithms for the RCP on a class of L-specified hierarchical planar graphs which we call Well-Separated (WS) graphs. In such graphs, levels in the hierarchy are allowed to directly connect *only* to their immediate descendants. In particular:

1. We prove that the decision version of the RCP for Well-Separated L-specified hierarchical planar graphs is $\mathcal{PSPACE}$-complete.

2. We present two approximation algorithms for RCP for this class of graphs. These algorithms offer alternative trade-offs between the quality and the efficiency of the solution achieved. The first one is a simple and very efficient 4-approximation algorithm, while the second one achieves a better solution; it is a 3-approximation algorithm, but is less efficient, although polynomial.

We note that the class of WS L-specified hierarchical graphs considered here can lead to graphs that are exponentially large in the size of their specification. The WS class is a subclass of the class of L-specified hierarchical graphs considered in [11], called k-level-restricted graphs.

1.3 Related Work and Comparison

In a fundamental work, Lengauer and Wagner [10] proved that the following problems are $\mathcal{PSPACE}$-complete for L-specified hierarchical graphs: 3-coloring, hamiltonian circuit and path, monotone circuit value, network flow and independent set. For L-specified graphs, Lengauer ([9]) have given efficient algorithms to solve several important graph theoretic problems including 2-coloring, min spanning forest and planarity testing.

Marathe et al in [12, 11] studied the complexity and provided approximation schemes for several graph theoretic problems for L-specified hierarchical planar graphs including maximum independent set, minimum vertex cover, minimum edge dominating set, max 3SAT and max cut.

We remark that the best currently known approximation ratio for the RCP on ordinary (non-hierarchical) planar graphs of general degree is 2 ([6, 1]). Approximations for various classes of graphs presented in [5]. Also, the only known results on any kind of coloring problems have been shown for the vertex coloring for a special kind of hierarchical graphs (k-level-restricted unit disk graphs) achieving a 6-approximation solution ([13]).

2 Preliminaries

In this work we study an optimization version of the Radiocoloring problem ([6]), where the objective is to minimize *the number* of colors used:

Definition 1. Min order RCP: *Given a graph $G(V, E)$, find an assignment of G, i.e. a coloring function $\Lambda : V \to N^*$ assigning integers (colors) to the vertices of G such that $\Lambda(u) \neq \Lambda(v)$ if $d(u, v) \leq 2$, where $d(u, v)$ is the distance between u and v in G, that uses a minimum number of colors. The number of different integers in such an assignment, is called the* order *of RCP on G and is denoted here by $\lambda(G)$, i.e. $\lambda(G) = |\Lambda(V)|$.*

For simplicity reasons, in the sequel we refer to it as the RCP. Remark that:

Proposition 1. *The min order RCP of a given graph G is equivalent to the problem of coloring the square of the graph G, G^2. G^2 has the same vertex set as G and there is an edge between any two vertices of G^2 if their distance in G is at most 2.*

We study the RCP on hierarchical graphs as specified by Lengauer [9].

Definition 2. (**L-specifications,** [9]) *An L-specification $\Gamma = (G_1, \cdots, G_i, \cdots, G_n)$, where n is the number of levels in the specification, of a graph G is a sequence of labeled undirected simple graphs G_i called* cells. *The graph G_i has m_i edges and n_i vertices. The p_i of the vertices are called pins. The other $(n_i - p_i)$ vertices are called inner vertices. The r_i of the inner vertices are called nonterminals. The $(n_i - r_i)$ vertices are called terminals. The remaining $n_i - p_i - r_i$ vertices of G_i that are neither pins nor nonterminals are called* explicit *vertices.*

Each pin of G_i has a unique label, its name. The pins are assumed to be numbered from 1 to p_i. Each nonterminal in G_i has two labels (v, t), a name and a type. The type t of a nonterminal in G_i is a symbol from $G_1, \cdots, G_{i-1}$. The neighbours of a nonterminal vertex must be terminals. If a nonterminal vertex v is of type G_j in G_i, $1 \leq j \leq i-1$, then v has degree p_j and each terminal vertex that is a neighbor of v has a distinct label (v, l) such that $1 \leq l \leq p_j$. We say that the neighbor of v labeled (v, l) matches the l-th pin of G_j.

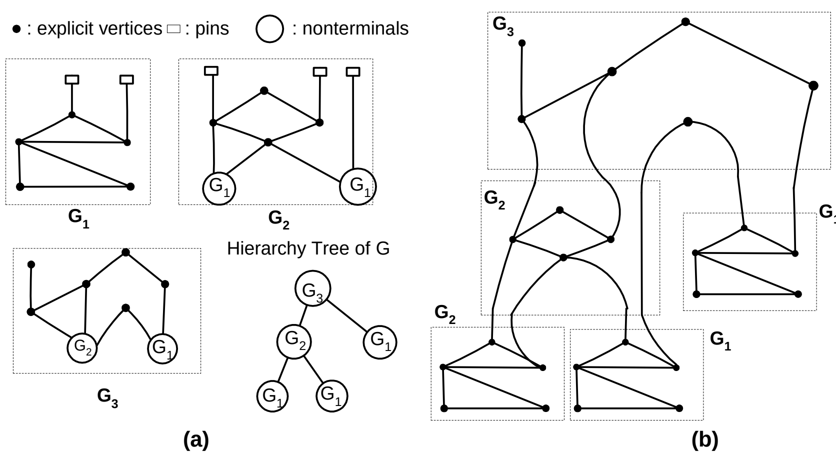

Fig. 1. **(a)** An L-specification $\Gamma = (G_1, G_2, G_3)$ of a graph G and its hierarchy tree $HT(G)$. **(b)** The expansion of the graph G.

See Figure 1(a) for an example of an L-specification. Note that a terminal vertex may be a neighbor of several nonterminal vertices. Given an L-specification Γ, $N = \sum_{1 \leq i \leq n} n_i$ denotes the *vertex number* and $M = \sum_{1 \leq i \leq n} m_i$ denotes the *edge number* of Γ. The size of Γ, denoted by $size(\Gamma)$, is $N + M$. For simplicity reasons, we assume that $n = \max_i \{n_i\}$.

Definition 3. (**Expansion of an L-specificied hierarchical graph,** [9]) *Let any L-specified hierarchical graph, given by $\Gamma = (G_1, \cdots, G_n)$. The* expanded graph *$E(\Gamma)$ (i.e. the graph associated with Γ) is iteratively obtained as follows:*

$k = 1$*:* $E(\Gamma) = G_1$.

$k > 1$*: Repeat the following step for each nonterminal* v *of* G_k *say of the type* G_j*: delete* v *and the edges incident on* v*. Insert a copy of* $E(\Gamma_j)$ *by identifying the* l*-th pin of* $E(\Gamma_j)$ *with the node in* G_k *that is labeled* (v, l)*. The inserted copy of* $E(\Gamma_j)$ *is called a subcell of* G_k*.*

For example, the expansion of the hierarchical graph G of Fig. 1(a) is shown in Fig. 1(b). To each L-specification $\Gamma = (G_1, \cdots, G_n)$, we associate a labeled rooted unoriented tree $HT(\Gamma)$ depicting the insertions of the copies of the graphs $E(\Gamma_j)$ $(1 \leq j \leq n-1)$, made during the construction of $E(\Gamma)$ as follows:

Definition 4. *(***Hierarchy Tree of an L-specification,** [**9**]*) Let* $\Gamma = (G_1, \cdots, G_n)$*, be an L-specification of the graph* $E(\Gamma)$*. The hierarchy tree of* Γ*, denoted by* $HT(\Gamma)$*, is a labeled rooted unordered tree defined as follows:*

1. *Let* r *the root of* $HT(\Gamma)$*. The label of* r *is* G_n*. The children of* r *in* $HT(\Gamma)$ *are in one-to-one correspondence with the nonterminal vertices of* G_n *as follows: The label of the child* s *of* r *in* $HT(\Gamma)$ *corresponding to the nonterminal vertex* (v, G_j) *of* G_n *is* (v, G_j)*.*
2. *For all other vertices* s *of* $HT(\Gamma)$ *and letting the label of* $s = (v, G_j)$*, the children of* s *in* $HT(\Gamma)$ *are in one-to-one correspondence with the nonterminal vertices of* G_j *as follows: The label of the child* t *of* s *in* $HT(\Gamma)$ *corresponding to the nonterminal vertex* (w, G_l) *of* G_j *is* (w, G_l)*.*

We consider hierarchical planar graphs as studied in [9]:

Definition 5. *(***Strongly planar hierarchical graph,** [**9**]**)** *An L-specified hierarchical graph* G *given by* $\Gamma = (G_1, \ldots, G_n)$ *is strongly planar if* $E(\Gamma)$ *has a planar embedding such that for each* $E(\Gamma_i)$ *all pins of it occur around a common face and the rest of* $E(\Gamma_i)$ *is completely inside this face.*

In fact, we here study a subclass of strongly planar hierarchical graphs, where additionally to the above condition, *all graphs* G_i*,* $1 \leq i \leq n$*, are planar.* We call this class as **fully planar hierarchical graphs**. In the sequel, and when there is no ambiguity, we refer to such graphs simply as *hierarchical planar graphs*.

Moreover, we concentrate on a class of L-specified hierarchical graphs which we call Well-Separated (WS) graphs, defined in the sequel using the followings:

Consider an L-specified hierarchical graph G, given by $\Gamma = (G_1, \ldots, G_n)$. For each graph G_i $(1 \leq i \leq n)$, we define the following subgraphs:

Definition 6. Inner subgraph of graph G_i, $\mathbf{G_{in\ i}}$*: is induced by the explicit vertices of* G_i *not connected to any pin or nonterminal of* G_i*.*

Definition 7. Outer-up subgraph of graph G_i, $\mathbf{G_{outUp\ i}}$*: is induced by the explicit vertices of* G_i *connected to at least one pin of* G_i*.*

Definition 8. Outer-down subgraph of graph G_i, $\mathbf{G_{outDown\ i}}$*: is induced by the explicit vertices of* G_i *connected to at least one nonterminal of* G_i*.*

Remark 1. Generally, an explicit vertex of G_i might belong to both outer-up, outer-down subgraphs of G_i. In this work we study the following class of graphs:

Definition 9. *(***Well-Separated, WS***) We call Well-Separated graphs the class of L-specified hierarchical graphs of which any explicit vertex of G_i, $1 \leq i \leq n$, belongs either to $G_{outDown\ i}$ or $G_{outUp\ i}$ or none of them, but not to both of them. Moreover, any vertex of $G_{outDown\ i}$ is located at distance at least 3 from any vertex of $G_{outUp\ i}$.*

Observe that the WS class of hierarchical graphs is testable in time polynomial in the size of the L-specification of a hierarchical graph. Note also that the WS class is a subclass of k-level-restricted graphs, which is another class of L-specified hierarchical graphs, studied by Marathe et al. [11] (see full version [2] for a proof).

Definition 10. *The* **maximum degree of a hierarchical graph** *G, $\Delta(G)$, is the maximum degree of a vertex in the expansion of the graph, $E(\Gamma)$.*

The following definitions and results are needed by our approximation algorithms.

Definition 11. *(***k-outerplanar graph** *G* **[3]***) A k-outerplanar graph G is defined recursively by taking an embedding of the planar graph G, finding the vertices in the exterior face of the graph and removing those vertices and the edges incident to them. Then, the remaining graph should be a $(k-1)$-outerplanar graph. A 1-outerplanar graph is an outerplanar graph.*

Theorem 1. *[4] Any k-outerplanar graph is a $3k-1$ bounded treewidth graph.*

Theorem 2. *[16] Let $G(V,E)$ be a k-tree of n vertices given by its tree-decomposition, let C be a set of colors, and let $\alpha = |C|$. Then, it can be determined in polynomial time $T(n,k)$, whether G has a radiocoloring that uses the colors of set C, and if such a radiocoloring exists, it can found in the same time, where $T(n,k) = O(n(2\alpha+1)^{2^{2(k+1)(l+2)+1}} + n^3)$, $l = 2$ and $n = |V|$.*

3 The Complexity of the Radiocoloring Problem

In this section, we study the complexity of RCP on L-specified hierarchical planar graphs. A critical observation about the constructions utilized in the $\mathcal{PSPACE}$-completeness proofs, is that the transformations are local ([7]). I.e, given any hierarchical graph G, the graph G_i' obtained from each G_i, is the same for all appearances of G_i in the hierarchy tree of G. Thus, the resulting hierarchical graph G' can be computed in time polynomial in the size of the L-specification of the graph G.

Another important issue for the $\mathcal{PSPACE}$-completeness reductions is whether an already known $\mathcal{NP}$-completeness proof for the same problem, that fulfills such locality characteristics, can be modified so that to apply for a hierarchical graph G. This technique has been used in previous papers to get $\mathcal{PSPACE}$-completeness results for a number of problems considered (e.g. [13]).

In our case, there was no such 'local' $\mathcal{NP}$-completeness reduction available. The corresponding $\mathcal{NP}$-completeness reductions that *both* could be adapted for the hierarchical case are the reductions of [15, 14]. However, although the reduction of [15] is local that of [14] is not. Henceforth, they can not be used to get the $\mathcal{PSPACE}$-completeness of L-specified hierarchical planar graphs.

For these reasons, we have developed a new $\mathcal{NP}$-completeness proof for the RCP of ordinary planar graphs which reduces it from the problem of 3-coloring planar graphs. The construction satisfies the desired locality characteristics and thus, we can utilize it to get the $\mathcal{PSPACE}$-completeness proof of the RCP for L-specified hierarchical planar graphs.

3.1 The $\mathcal{NP}$-completeness of RCP for planar graphs

In this section we provide a new $\mathcal{NP}$-completeness proof for the problem of radiocoloring for ordinary (non-hierarchical) planar graphs, which is 'local'. We remark that this reduction is the only one that works for the cases where $\Delta(G) < 7$ ($\Delta(G) \geq 3$), in contrast to the only known $\mathcal{NP}$-completeness proof of [15].

Theorem 3. *The following decision problem is $\mathcal{NP}$-complete:*
Input: A planar graph $G(V, E)$.
Question: Is $\lambda(G) \leq 4$?

Proof. It can be easily seen that the problem belongs in $\mathcal{NP}$. Let any planar graph G. We reduce RCP from the 3-COLORING problem of planar graphs, which is known to be $\mathcal{NP}$-complete ([7]). I.e. we will construct in polynomial time a new graph G', which is 4-radiocolorable if and only if G is 3-colorable. The reduction employs the component design technique.

The construction replaces every vertex u of degree d_u of the initial graph G with a component, called 'cycle node'. The cycle node obtained by a vertex of degree d_u is said to be 'a cycle node of degree d_u'. An instance of it is shown in Figure 2(a). A cycle node of degree d_u is constructed as follows:

Add a cycle of $3d_u$ vertices, called *outer cycle*, as shown in Figure 2(a). Call the vertices of each triad as *first, second* and *third.* For each triad, add two more vertices (called *fourth* and *fifth*) and connect them to the triad as shown in the Figure. Now, group together every five such vertices into pentads and number them as shown in the Figure. Next, add another cycle of $3d_u$ vertices, called *inner cycle.* For each triad, add a fourth vertex and connect it to the triad as shown in the Figure. Now, group together every four such vertices into quadruples and call them as in Figure 2(a). Finally, connect the i-th pentad of the outer cycle to the i-th quadruplet of the inner cycle as in the Figure. In the sequel, we explain how the cycle node is used to construct in polynomial time from any planar graph G a new planar graph G', with the desired properties. We consider a planar embedding of graph G. The new graph G' is constructed as follows (See Figure 2(b) for an example):

1. Replace each vertex of degree d_u of the initial graph G, with a cycle node of degree d_u.

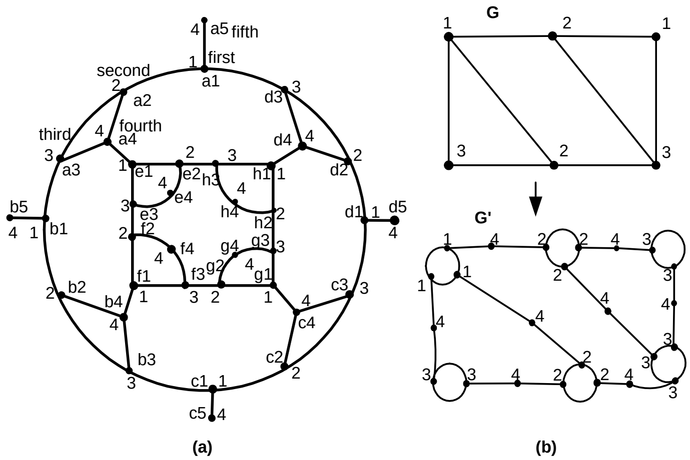

Fig. 2. (a) The 'cycle node' for a vertex of degree 4 and a 4-radiocoloring of it. (b) A graph G with a 3-coloring and the graph G' obtained with the resp. 4-radiocoloring.

2. For each vertex u of graph G, number the edges incident to u, in increasing, clockwise, order. For every edge of the initial graph $e = (u, v)$ connecting u and v, let u_i be the number of edge e given by vertex u and let v_j be the number of the edge e given by vertex v. Then, take the fifth vertex of the u_i-th group of the cycle node of vertex u and the fifth vertex of the v_j-th group of the cycle node of vertex v and collapse them to a single vertex uv.

Obviously, the new graph G' is a planar graph.

Lemma 1. $\lambda(G') \leq 4$ *if and only if* $\chi(G) \leq 3$.

(See the full version [2] of the paper for the proof of the Lemma) □

3.2 The $\mathcal{PSPACE}$-completeness of RCP for Hierarchical Planar Graphs

The $\mathcal{PSPACE}$-completeness reduction for the RCP of L-specified hierarchical planar graphs, utilizes the 'local' construction of the $\mathcal{NP}$-completeness proof of the radiocoloring problem for ordinary planar graphs, given in section 3.1, Theorem 3, in order to be of polynomial time to the size of the L-specification.

The $\mathcal{PSPACE}$-completeness of 3-coloring for Hierarchical Planar Graphs

In order to be able to utilize the $\mathcal{NP}$-completeness reduction of Theorem 3 to prove the $\mathcal{PSPACE}$-completeness of RCP for hierarchical planar graphs, we need to prove that:

Theorem 4. *The following decision problem is $\mathcal{PSPACE}$-complete:*

Input: A fully planar hierarchical graph G, given by the L-specification $\Gamma = (G_1, \ldots, G_n)$.

Question: Is $\chi(G) \leq 3$?

Proof. We adapt the $\mathcal{NP}$-completeness construction of [7]. See full version of the paper [2] for the proof of the Theorem. □

The *PSPACE*-completeness of RCP for hierarchical planar graphs Before introducing the $\mathcal{PSPACE}$-completeness Theorem, the following observation is needed. We denote by $d(u)_H$ the degree of vertex u in graph H. Observe that, for an L-specified hierarchical graph G, for any G_i, $i = 1, \ldots n$, $d(u)_{G_i} = d(u)_{E(\Gamma)}$, $u \in G_i$.

Theorem 5. *The following decision problem is $\mathcal{PSPACE}$-complete:*

Input: A WS fully planar hierarchical graph G, given by the L-specification $\Gamma = (G_1, \ldots, G_n)$.

Question: Is $\lambda(G) \leq 4$?

Proof. **Membership in $\mathcal{PSPACE}$:** Similar to the $\mathcal{PSPACE}$-membership of 3-coloring an L-specified hierarchical graph ([10]).

$\mathcal{PSPACE}$-completeness: We reduce the RCP for WS fully planar hierarchical graphs from the 3-COLORING of fully planar hierarchical graphs, proved to be $\mathcal{PSPACE}$-complete in Theorem 4 using the construction of Theorem 3.

Let any L-specified hierarchical planar graph G, given by $\Gamma = (G_1, \ldots, G_n)$. For each graph $G_i = (V_i, E_i)$ of its L-specification we construct a new graph $G'_i = (V'_i, E'_i)$ using the rules of Theorem 3:

1. Apply Rule 1 of Theorem 3 on the explicit vertices of G_i. Apply Rules 2,3 of the Theorem on the edges of G_i connecting any two explicit vertices of it.
2. All nonterminal vertices of G_i are present in G'_i. However, the type of a nonterminal, assume one of type G_j is changed to G'_j.
3. Take each $e = (u, t)$ of G_i connecting an explicit vertex u to a nonterminal vertex t of type G_j, matching the l-th pin of the graph G_j. Then, connect the corresponding fifth vertex of the cycle node of u to the nonterminal t matching the l-th pin of G_j.
4. All pins vertices of G_i are present in G'_i. Take each edge $e = (u, p)$ of G_i connecting an explicit vertex u to a pin p of G_i, numbered as the l-th. Then, remove the corresponding fifth vertex of the cycle node of vertex u and connect the corresponding first vertex of the cycle node to the pin p.

Observe that graphs $G'_1, G'_2, \ldots, G'_n$ obtained G, define an L-specified hierarchical graph G', given by $\Gamma' = (G'_1, G'_2, \ldots, G'_n)$. To see why note that the graph G'_i obtained by G_i is the same for all appearances of G_i in the hierarchical tree $HT(\Gamma)$ of G. Moreover, the graph G'_i has the same pins and calls the same terminals as the initial G_i.

Lemma 2. $\lambda(G') \leq 4$ *if and only if* $\chi(G) \leq 3$. □

The lemma is proved using the same arguments as those in the proof of Lemma 1 and the observation that the expansion of the graph G' obtained, $E(\Gamma')$, is the same graph as the graph obtained by the construction of Theorem 3 when applied on the expansion of the initial hierarchical graph G.

Finally, it can be easily proved (see full version [2]) that the resulting graph is a WS fully planar hierarchical graph. □

4 Approximations to RCP for WS Fully Planar Graphs

In this section we present two approximation algorithms for the min order RCP on WS fully planar hierarchical graphs: a simple and fast algorithm, that achieves an approximation ratio of 4, and a more sophisticated one which, being still polynomial, achieves a 3-approximation ratio. These algorithms offer alternative options that trade-off the efficiency of the algorithm and the quality of the solution achieved. Both algorithms utilize a bottom up methodology of radiocoloring an L-specified hierarchical planar graph G, given by $\Gamma = (G_1, \ldots, G_n)$. Actually, they compute at most $n-i$ radiocolorings for a subgraph of each graph G_i, $1 \leq i \leq n$, and use these radiocolorings for all copies of G_i in the expansion of G, $E(\Gamma)$. This enables them to run in time only polynomial to the size of the L-specification of G.

More analytically, we wish to compute only one radiocoloring assignment for each G_i, and use this in all appearances of G_i in the expansion of G, $E(\Gamma)$. However, due to the structure of L-specified hierarchical graphs, the distance two neighborhood of the outer vertices of each G_i, may differentiate for every call of G_i by other graphs G_j. Henceforth, a radiocoloring for such a vertex (the outer ones) may becomes invalid due to a change of the distance two neighborhood of the vertex. Since each graph G_i may be called by at most $n-i$ other graphs, we need to compute at most $n-i$ radiocolorings of those (outer) vertices.

Moreover, we need to guarantee that the different radiocolorings of the outer part of G_i do not introduce any implication in the radiocoloring of its inner part. By having only one radiocoloring for the inner part of G_i, we manage to have also no implications to the radiocoloring of the subtree of G_i, $HT(G_i)$.

Based on this design approach, both algorithms partition appropriately, each G_i into three parts: (1) *the inner part*, (2) *the outer up* and (2) *the outer down part*. Remark, that these subgraphs might be different from the inner, outer up, outer down subgraphs of G_i defined in the Definitions 6, 7 and 8.

Then, the algorithms radiocolor the inner part of G_i only once, using, each of them, a different method. Both of them, they group and radiocolor the outer down part of it together with the outer up parts of the graphs called by it, using a known 2-approximation algorithm.

4.1 A 4-approximation Algorithm HRC_1

We first provide a simple and efficient algorithm (HRC_1) that achieves a 4-approximation for RCP on WS fully planar hierarchical graphs. Let A_1, B_1 two disjoint sets of colors of size $2\Delta(G) + 25$ each, where $\Delta(G)$ is the maximum degree of G.

Overview of the HRC_1 Algorithm: First, the algorithm defines for each G_i its inner, outer up and outer down parts to be the inner, outer up and outer down subgraphs $G_{in\ i}$, $G_{outUp\ i}$ and $G_{outDown\ i}$, respectively.

Then, it radiocolors the inner part of the graph G_i using a known 2-approximation algorithm e.g. [6, 1] using the color set A_1. Also, it radiocolors the outer down part of the graph G_i together with the outer up parts of its children using the 2-approximation algorithm with the color set B_1.

Theorem 6. *Algorithm $HRC_1(G)$ produces a radiocoloring of a WS fully planar hierarchical graph G in time $O(n^5)$ and achieves a 4-approximation ratio.*

For, a detailed description of the HRC_1 algorithm and the proof of Theorem 6, see the full version of the paper [2]. □

4.2 A 3-approximation Algorithm HRC_2

Overview of the Algorithm: We provide a more sophisticated radiocoloring algorithm, that achieves a 3-approximation ratio rather than 4, for fully planar hierarchical graphs of class WS. The basic idea of the algorithm, called HRC_2, is to partition the vertices of each graph G_i into outerplanar *levels* using a BFS (similar to [3, 8]) and define the three parts of each G_i based on this search.

The *outer up part of* G_i consists of the first level of the BFS tree obtained. Thus, it is the outer up subgraph of G_i, $G_{outUp\ i}$. The *outer down part of* G_i consists of the graph induced by the vertices of the BFS tree of levels D up to the end of the tree, where D is the first level of the tree having an outer down vertex of G_i. The *inner part of* G_i *is the rest of the graph* G_i.

More analytically, the *inner part of* G_i *is radiocolored* as follows: Radiocolor every two successive levels of the inner part of G_i optimally interchanging color sets A, B, where $|A| = |B| = OPT(G)$ and $OPT(G)$ is the optimal number of colors needed to radiocolor G. This can be achieved in polynomial time and without any conflicts as we prove: We first show that any two successive levels, call them a double level, is a 4-outerplanar graph. Thus, by Theorem 1, it is a bounded treewidth graph. Consequently, applying Theorem 2, each double level can be radiocolored optimally in polynomial time. Moreover, by the BFS partitioning procedure, there is no conflict between double levels colored using the same color set.

The *outer down part of* G_i *is radiocolored* together with the outer up parts of its children using a known (2-approximation) radiocoloring algorithm for ordinary planar graphs using color sets A and C, where $|C| = |A| = OPT(G)$. Since the algorithm uses only color sets A, B, C, it has a 3-approximation algorithm.

Theorem 7. *HRC_2 Algorithm produces a valid radiocoloring of a WS fully planar hierarchical graph G using at most $3\ OPT(G) + 50$ colors, achieving a 3 approximation ratio. It runs in $O(n^2 \cdot T(n,k) + n^5)$ time, where $T(n,k)$ is a polynomial time function for the optimal radiocoloring of a k-tree of size n, specified in [16] ($k = 11$), (see Theorem 2).*

For, a detailed description of the HRC_2 algorithm and the proof of Theorem 7, see the full version of the paper [2].

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