A Graph-Theoretic Network Security Game*

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Abstract

Consider a network vulnerable to viral infection, where the security software can guarantee safety only to a limited part of it. We model this practical network scenario as a non-cooperative multi-player game on a graph, with two kinds of players, a set of *attackers* and a *protector* player, representing the viruses and the system security software, respectively. Each attacker player chooses a node of the graph via a probability distribution to infect. The protector player chooses either an edge or a simple path of the network and cleans this part from attackers. Each attacker wishes to maximize the probability of escaping its cleaning by the protector. In contrast, the protector aims at maximizing the expected number of extinguished attackers. We call the two games obtained the *Path* and the *Edge* model, respectively.

We are interested in the associated *Nash equilibria*, where no network entity can unilaterally improve its local objective. We obtain the following results:

- For certain families of graphs, mixed Nash equilibria can be computed in polynomially time. These families include, among others, regular graphs, graphs with perfect matchings and trees.
- The corresponding *Price of Anarchy* for any mixed Nash equilibria of the game is upper and lower bounded by a linear function of the number of vertices of the graph. (We define the Price of Anarchy to reflect the utility of the protector.)
- The problem of existence of a pure Nash equilibrium for the Path model is $\mathcal{NP}$-complete.

1 Introduction

1.1 Motivation

This work considers a problem of *Network Security* related to the protection of a system from harmful procedures (e.g., viruses, worms). Consider an information network where the nodes of the network are insecure and vulnerable to infection from entities called *attackers*, such as viruses and Trojan horses. A *protector*, such as the system security software, is available to the system, but it can guarantee security only to a limited part of the network, such as a simple path or a single link of it. Each harmful entity targets a location (such as a node) of the network; the node is damaged unless it is cleaned by the system security software.

Apparently, the harmful entities and the system security software have conflicting objectives. The security software seeks to protect the network as much as possible, while the harmful entities wish to avoid being caught by the software so that they be able to damage the network. Thus, the system security software seeks to maximize the expected number of viruses it catches, while each harmful entity seeks to maximize the probability it escapes from the security software.

Naturally, we model this scenario as a *non-cooperative strategic game* played on a graph with two kinds of players: the *vertex players* representing the harmful entities, and the *edge* or the *path player* representing the system security software considered in each of the two cases, choosing a single edge or a simple path, respectively. The corresponding games are called the ***Edge*** and the ***Path*** model, respectively. In each model, each player seeks to maximize her *Individual Profit*. We are interested in the *Nash equilibria* [12, 13] associated with these games, where no player can unilaterally improve its Individual Profit by switching to a more advantageous probability distribution.

We measure the system performance utilizing *Social Cost* [5], defined as the number of attackers caught by the protector.

1.2 Contribution

Our results are summarized as follows:

1.2.1 The Edge Model

r-Factorizable, Regular and Perfect Matching Graphs.

Although [8] provides a graph-theoretic characterization of mixed Nash Equilibria for the Edge model, the characterization only implies an exponential time algorithm for the case of general graphs. Here, we utilize the characterization to provide polynomial time algorithms to compute mixed Nash equilibria for *specific* graphs. In particular, we combine the characterization with a suitable exploration of some graph-theoretic properties of each graph family considered to obtain polynomial time *structured* mixed Nash equilibria.

We consider *r-factorizable* graphs, which contain a spanning r-regular subgraph, for some positive integer r. We show that an r-factorizable graph G admits a mixed Nash equilibrium that can be computed in time $O(T(G))$, where $O(T(G))$ is the time needed for the computation of an r-factor of G (Theorem 3.1). So, if $T(G)$ is a polynomial function, a Nash equilibrium can be computed in polynomial time (Corollary 3.2). Also, this implies that *regular* graphs admit polynomially computable Nash equilibria (Corollary 3.3). Furthermore, since a graph with a Perfect Matching (called *Perfect Matching graph*) is an 1-factorizable graph and a Perfect Matching can be computed in polynomial time [3], the same result implies that Perfect Matching graphs admit polynomial time Nash equilibria (Corollary 3.4).

Trees.

We present a *linear* time algorithm for computing a mixed Nash equilibrium for a tree (Theorem 3.5). This improves an algorithm to compute such an equilibrium that runs in time $O\left(\sqrt{|V|}|E|\cdot \log_{|V|}\frac{|V|^2}{|E|}\right)$ from [10].

Price of Anarchy.

We present upper and lower bounds on the Social Cost of any mixed Nash equilibrium in the Edge model (Theorem 3.10). Using these bounds, we show that the corresponding Price of Anarchy is upper and lower bounded by a linear function of the number of vertices of the graph (Theorem 3.12).

1.2.2 The Path Model

We prove that the existence problem of pure Nash equilibria in the Path model is $\mathcal{NP}$-complete (Theorem 4.4). This is in contrast to the Edge model, where no instance has a pure Nash equilibrium [8, Theorem 1].

1.3 Related Work and Significance

This work is a step further in the development of *Algorithmic Game Theory.* It is also one of the very few works modeling network security problems using strategic games. Such a research line is that of *Interdependent Security* games [1]. However, *none* of these works, with an exception of [1], studied the associated Nash equilibria. In [1], the authors studied a particular Virus Inoculation game and established connections with variants of the Graph Partition problem.

The Edge model was introduced and studied in [8], where a non-existence result for pure Nash equilibria (for any instance) and a polynomial time algorithm to computed mixed Nash equilibria for bipartite graphs were provided.

[10] provided a polynomial time characterization of graphs admitting a class of *structured* Nash equilibrium the so called *Matching Nash equilibria*. The characterization implies that trees admit such equilibria. They also presented an algorithm to compute such them (if such one exists) for a graph that runs in time $O\left(\sqrt{|V|}|E|\cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$. Here, we present a more efficient, *linear* time algorithm for computing a *structured* Nash equilibrium for a tree. More recently, [9] showed that a more general class of graphs admits polynomial time computable Nash equilibria; the graphs with Fractional Perfect Matchings.

Recently, generalizations of the Edge model have been introduced and investigated:

- In [4], the protector is able to scan and protect a set of k links of the network. That work presented a polynomial time algorithm for computing pure and mixed Nash equilibria and a polynomial-time transformation of a special class of structured Nash equilibria between the Edge model and the generalized model.
- In the generalization considered in [7], there is a number of *interdependent* protectors with a *reward-sharing* scheme. The results in [7] provide interesting trade-offs between the number of the protectors and the amount of protection.

2 Framework

Throughout, we consider an undirected graph $G = G(V, E)$ with no isolated vertices; G is *non-trivial* whenever it has more than one edges, otherwise it is *trivial*.

2.1 Graph Theory

For a vertex set $U \subseteq V$, denote $G(U)$ the subgraph of G induced by U; denote $\mathsf{Edges}_G(U) = \{(u,v) \in E \mid \ u,v \in U\}$. For an edge set $E' \subseteq E$, denote $\mathsf{Vertices}_G(E') = \{v \in V \mid (v,u) \in E'\}$. For a vertex set $U \subseteq V$, denote $\mathsf{Neigh}_G(U) = \{u \notin U \mid (u,v) \in E \text{ for some vertex } v \in U\}$. For an edge set $F \subseteq E$, denote $G(F)$ the subgraph of G induced by F. For a vertex set $U \subseteq V$, the graph $G \backslash U$ is obtained by by G by removing the vertices of set U and their incident edges. For a vertex $v \in V$, denote $\Delta_G(v)$ the degree of vertex v and $\Delta(G) = \max_{v \in V} \Delta_G(v)$.

- An ***Independent Set*** is a vertex set $IS \subseteq V$ such that for all pairs of vertices $u, v \in IS$, $(u, v) \notin E$.

- A ***Vertex Cover*** is a vertex set $VC \subseteq V$ such that for each edge $(u, v) \in E$ either $u \in VC$ or $v \in VC$.

- An ***Edge Cover*** is an edge set $EC \subseteq E$ such that for every vertex $v \in V$, there is an edge $(v, u) \in EC$.

- A ***Matching*** is a set $M \subseteq E$ of non-incident edges. A *Maximum Matching* is one that has maximum size; $\alpha'(G)$ denotes the size of a Maximum Matching and it is called the ***Matching Number***. The currently fastest algorithm to compute a Maximum Matching of G appears in [3] and has running time $O\left(\sqrt{|V|}|E| \cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$. It is known that a Minimum Edge Cover can be computed in polynomial time via computing a Maximum Matching. (See, e.g., [15, page 115].) A ***Perfect Matching*** is a Matching that is also an Edge Cover. A graph that admits a Perfect matching is called ***Perfect Matching*** graph.

- A (simple) *path* is a sequence $v_1, v_2, \cdots, v_k$ of distinct vertices from V such that $v_i, v_{i+1} \in E$ for $1 \leq i < k$; say that the path has *size* k. A ***Hamiltonian path*** is a path that includes all vertices from V. A graph containing a Hamiltonian path is called *Hamiltonian*. It is well known that answering the question whether a given graph is Hamiltonian is an $\mathcal{NP}$-complete [2, problem GT39]. Denote $\mathsf{Paths}(G)$ the set of all possible paths in G.

- For a ***rooted*** tree graph $T = (V, E)$ denote $root \in V$, the root of the tree and $\mathsf{leaves}(T)$ the leaves of the tree T. For any $v \in V$, denote $\mathsf{parent}_T(v)$ the parent of v in the tree and $\mathsf{children}_T(v)$ its children in the tree T. For any $A \subseteq V$, let $\mathsf{parents}_T(A) := \{u \in V \mid u = \mathsf{parent}_T(v),\ v \in A\}$.

- For an integer r, graph G is ***r-regular*** if $\Delta_G(v) = r$, $\forall v \in V$. A *r-factor* of a graph G, is a spanning subgraph of G such that $\Delta_{G_r}(v) = r$ for any $v \in V$. We denote it as $G_r = (V, E')$, where $E' \subset E$. A graph containing an r-factor is called ***r-factorizable***. For an 1-factor of G, its edge set is a Perfect Matching of G. A 2-factor of a graph, also called *cycle cover*, (if there exists) can be computed in polynomial time, via Tutte's reduction [14] to the classical Perfect Matching problem (see [6, Sect. 10.1]). Thus, the currently most efficient algorithm for its computation needs time $O\left(\sqrt{|V|}|E| \cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$ [3].

  A graph G is called ***polynomially computable r-factorizable*** if it contains a an r-factor subgraph that can be compute in polynomial time. It can be easily seen that there exist exponential many such graph instances.

2.2 The Edge Model

Associated with G is a ***strategic game*** $\Pi_{\mathsf{E}}(G) = \langle \mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{\mathsf{IP}\}_{i \in \mathcal{N}} \rangle$ on G:

- The set of ***players*** is $\mathcal{N} = \mathcal{N}_{vp} \cup \mathcal{N}_{ep}$, where $\mathcal{N}_{vp}$ contains ν ***vertex players*** vp_i, $1 \leq i \leq \nu$ and $\mathcal{N}_{ep}$ contains a single ***edge player*** ep.
- The ***strategy set*** S_i of vertex player vp_i is V; the ***strategy set*** S_{ep} of the edge player ep is E. So, the ***strategy space*** $\mathcal{S}$ of the game is $\mathcal{S} = \left(\times_{i \in N_{vp}} S_i \right) \times S_{ep} = V^{\nu} \times E$.
- Fix any ***profile*** $\mathbf{s} = \langle s_1, \ldots, s_\nu, s_{ep} \rangle \in \mathcal{S}$, also called a ***pure profile***:
  - The ***Individual Profit*** of vertex player vp_i is a function $\mathsf{IP}_i(\mathbf{s}) : \mathcal{S} \to \{0, 1\}$ such that $\mathsf{IP}_i(\mathbf{s}) = \begin{cases} 0, & s_i \in s_{ep} \\ 1, & s_i \notin s_{ep} \end{cases}$; so, the vertex player vp_i receives 1 if it is not caught by the edge player, and 0 otherwise.
  - The ***Individual Profit*** of the edge player ep is a function $\mathsf{IP}_{ep}(\mathbf{s}) : \mathcal{S} \to \mathbb{N}$ such that $\mathsf{IP}_{ep}(\mathbf{s}) = |\{i : s_i \in s_{ep}\}|$; so, the edge player ep receives the number of vertex players it catches.

2.2.1 Pure Strategies and Pure Nash Equilibria

The profile $\mathbf{s}$ is a ***pure Nash equilibrium*** [12, 13] if for each player $i \in \mathcal{N}$, it maximizes $\mathsf{IP}_i(\mathbf{s})$ over all profiles $\mathbf{t}$ that differ from $\mathbf{s}$ only with respect to the strategy of player i. Intuitively, in a pure Nash equilibrium, vertex player (resp., the edge player) can (resp., cannot) improve its Individual Profit by switching to a different vertex (resp., edge). In other words, a pure Nash equilibrium is a *local maximizer* for the Individual Profit of each player. Say that G *admits* a pure Nash equilibrium if there is a pure Nash equilibrium for the strategic game $\Pi_{\mathsf{E}}(G)$.

2.2.2 Mixed Strategies and Profiles

A ***mixed strategy*** for player $i \in \mathcal{N}$ is a probability distribution over S_i; thus, a mixed strategy for a vertex player (resp., edge player) is a probability distribution over vertices (resp., edges). A ***mixed profile*** $\mathbf{s} = \langle s_1, \ldots, s_\nu, s_{ep} \rangle$ is a collection of mixed strategies; $s_i(v)$ is the probability that vertex player vp_i chooses vertex v and $s_{ep}(e)$ is the probability that the edge player ep chooses edge e.

The ***support*** of player $i \in \mathcal{N}$ in the profile $\mathbf{s}$, denoted $\mathsf{Support}_i(\mathbf{s})$, is the set of pure strategies in its strategy set to which i assigns a strictly positive probability in $\mathbf{s}$. Denote $\mathsf{Support}_{vp}(\mathbf{s}) = \bigcup_{i \in \mathcal{N}_{vp}} \mathsf{Support}_i(\mathbf{s})$. Set $\mathsf{Edges}_v(\mathbf{s}) = \big\{(u,v) \in E : (u,v) \in \mathsf{Support}_{ep}(\mathbf{s})\big\}$. So, $\mathsf{Edges}_v(\mathbf{s})$ contains all edges incident to v that are included in the support of the edge player. For a vertex set $U \subseteq V$, set $\mathsf{Edges}_U(\mathbf{s}) = \{e = (u,v) \in \mathsf{Support}_{ep}(\mathbf{s}) : u \in U\}$. So, $\mathsf{Edges}_U(\mathbf{s})$ contains all edges incident to a vertex in U that are included in the support of the edge player.

For a vertex $v \in V$, the probability the edge player ep chooses an edge that contains the vertex v is denoted $P_{\mathbf{s}}(\mathsf{Hit}(v))$. Clearly, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges}_v(\mathbf{s})} s_{ep}(e)$. For a vertex $v \in V$, denote as $\mathsf{VP}_v(\mathbf{s})$ the expected number of vertex players choosing vertex v according to $\mathbf{s}$; so, $\mathsf{VP}_v(\mathbf{s}) = \sum_{i \in \mathcal{N}_{vp}} s_i(v)$. For each edge $e = (u,v) \in E$, $\mathsf{VP}_e(\mathbf{s})$ is the expected number of vertex players choosing either the vertex u or the vertex v.

2.2.3 Expected Individual Profit and Conditional Expected Individual Profits

A mixed profile $\mathbf{s}$ induces an ***Expected Individual Profit*** $\mathsf{IP}_i(\mathbf{s})$ for each player $i \in \mathcal{N}$, which is the expectation according to $\mathbf{s}$ of the Individual Profit of player i.

Induced by the mixed profile $\mathbf{s}$ is also the ***Conditional Expected Individual Profit*** $\mathsf{IP}_i((\mathbf{s}_{-i}, v))$ of vertex player $vp_i \in \mathcal{N}_{vp}$ on vertex $v \in V$, which is the conditional expectation according to $\mathbf{s}$ of the Individual Profit of player vp_i had it chosen vertex v. So,

$$\begin{aligned}\mathsf{IP}_i((\mathbf{s}_{-i}, v)) &= 1 - P_{\mathbf{s}}(\mathsf{Hit}(v)) \\ &= 1 - \sum_{e \in \mathsf{Edges}_v(\mathbf{s})} s_{ep}(e)\end{aligned}$$

Clearly, for the vertex player $vp_i \in \mathcal{N}_{vp}$,

$$\begin{aligned}\mathsf{IP}_i(\mathbf{s}) &= \sum_{v \in V} s_i(v) \cdot \mathsf{IP}_i((\mathbf{s}_{-i}, v)) \\ &= \sum_{v \in V} s_i(v) \cdot \left(1 - \sum_{e \in \mathsf{Edges}_v(\mathbf{s})} s_{ep}(e)\right).\end{aligned}$$

Finally, induced by the mixed profile $\mathbf{s}$ is the ***Conditional Expected Individual Profit*** $\mathsf{IP}_{ep}((\mathbf{s}_{-ep}, e))$ of the edge player ep on edge $e = (u,v) \in E$, which is the conditional expectation according to $\mathbf{s}$ of the Individual Profit of player ep had it chosen edge e. So,

$$\begin{aligned}\mathsf{IP}_{ep}((\mathbf{s}_{-ep}, e)) &= \mathsf{VP}_e(\mathbf{s}) \\ &= \sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v)).\end{aligned}$$

Clearly, for the edge player ep,

$$\begin{aligned}\mathsf{IP}_{ep}(\mathbf{s}) &= \sum_{e \in E} s_{ep}(e) \cdot \mathsf{IP}_{ep}((\mathbf{s}_{-ep}, e)) \\ &= \sum_{e=(u,v) \in E} s_{ep}(e) \cdot \left(\sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v)) \right).\end{aligned}$$

2.2.4 Mixed Nash Equilibria

The mixed profile $\mathbf{s}$ is a ***mixed Nash equilibrium*** [12, 13] if for each player $i \in \mathcal{N}$, it maximizes $\mathsf{IP}_i(\mathbf{s})$ over all mixed profiles $\mathbf{t}$ that differ from $\mathbf{s}$ only with respect to the mixed strategy of player i. In other words, a Nash equilibrium $\mathbf{s}$ is a *local maximizer* for the Expected Individual Profit of each player. By Nash's celebrated result [12, 13], there is at least one mixed Nash equilibrium for the strategic game $\Pi_{\mathsf{E}}(G)$; so, every graph G admits a mixed Nash equilibrium.

The particular definition of Expected Individual Profits implies in a Nash equilibrium, for each player $i \in \mathcal{N}$ and strategy $x \in S_i$ such that $s_i(x) > 0$, all Conditional Expected Individual Profits $\mathsf{IP}_i((\mathbf{s}_{-i}, x))$ are the same and no less than any Conditional Expected Individual Profit $\mathsf{IP}_i((\mathbf{s}_{-i}, x'))$ with $s_i(x') = 0$, where $x' \in S_i$. It follows that:

- For each vertex player vp_i, for any vertex $v \in \mathsf{Support}_i(\mathbf{s})$,

$$\mathsf{IP}_i(\mathbf{s}) = 1 - \sum_{e \in \mathsf{Edges}_v(\mathbf{s})} s_{ep}(e).$$

- For the edge player ep, for any edge $(u, v) \in \mathsf{Support}_{ep}(\mathbf{s})$,

$$\mathsf{IP}_{ep}(\mathbf{s}) = \sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v)).$$

Note that for each vertex player vp_i, there is some vertex v such that $s_i(v) > 0$; since a Nash equilibrium $\mathbf{s}$ maximizes the Individual Profit of the edge player ep, it follows that $\mathsf{IP}_{ep}(\mathbf{s}) > 0$ for a Nash equilibrium $\mathbf{s}$. A profile $\mathbf{s}$ is ***uniform*** if each player uses a uniform probability distribution on her support.

2.2.5 Background

In [8, Theorem 1] it was proved that if G contains more than one edges, then $\Pi_{\mathsf{E}}(G)$ has no pure Nash Equilibrium. We use a characterization of Nash equilibrium proved in the same work:

Theorem 2.1 ([8]) *A profile* $\mathbf{s}$ *is a Nash equilibrium if and only if* (1) *for each vertex* $v \in \mathsf{Support}_{vp}(\mathbf{s})$, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \min_{v' \in V} P_{\mathbf{s}}(\mathsf{Hit}(v'))$, *and* (2) *for each edge* $e \in \mathsf{Support}_{ep}(\mathbf{s})$, $\mathsf{VP}_e(\mathbf{s}) = \max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s})$.

We also use the notion of ***Covering profiles***, introduced in [8]. A covering profile is a profile $\mathbf{s}$ such that (1) $\mathsf{Support}_{ep}(\mathbf{s})$ is an Edge Cover of G and (2) $\mathsf{Support}_{vp}(\mathbf{s})$ is a Vertex Cover of the graph $G(\mathsf{Support}_{ep}(\mathbf{s}))$. It turns out that Covering profiles are interesting:

Proposition 2.2 ([8]) *A Nash Equilibrium is a Covering profile.*

An ***Independent Covering profile*** [8] $\mathbf{s}$ is a uniform Covering profile in which $s_i = s_j$, for any two vertex players $i, j \in \mathcal{N}_{vp}$ such that (1) $\mathsf{Support}_{vp}(\mathbf{s})$ is an Independent Set and (2) each vertex in $\mathsf{Support}_{vp}(\mathbf{s})$ is incident to exactly one edge in $\mathsf{Support}_{ep}(\mathbf{s})$. Our analysis uses the following previous result:

Theorem 2.3 ([10]) *An Independent Covering profile is a Nash equilibrium.*

2.3 The Path Model

We now introduce a generalization of the Edge model $\Pi_{\mathsf{E}}(G)$. The generalization consists of allowing the prodector (edge player in the Edge model) to select a path of G, instead of a single edge. We call this generalization as the ***Path model*** and denote it as $\Pi_{\mathsf{P}}(G) = \langle \mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{\mathsf{IP}\}_{i \in \mathcal{N}} \rangle$. The only difference with $\Pi_{\mathsf{E}}(G)$ is that:

(i) $\mathcal{N} = \mathcal{N}_{vp} \cup \mathcal{N}_{pp}$, where $\mathcal{N}_{vp}$ is defined as before but $\mathcal{N}_{pp}$ contains a single *path* player pp (which replaces the edge player)

(ii) $s_{pp} = \mathsf{Paths}(G)$.

Thus, the ***strategy set*** $\mathcal{S}$ of $\Pi_{\mathsf{P}}(G)$ is $\mathcal{S} = \left(\times_{i \in N_{vp}} S_i \right) \times S_{pp} = V^{\nu} \times |\mathsf{Paths}(G)|$.

Note that the strategy set of $\Pi_{\mathsf{P}}(G)$ is *exponential*, as the number of distinct paths of the graph.

2.3.1 Nash Equilibria

In the same way as for $\Pi_{\mathsf{E}}(G)$, we define a ***pure Nash equilibrium*** for $\Pi_{\mathsf{P}}(G)$. Similarly, we define a ***mixed strategy*** s_i for player $i \in \mathcal{N}$ and a ***mixed profile*** $\mathbf{s}$ for $\Pi_{\mathsf{P}}(G)$. In a mixed profile $\mathbf{s}$, we define the support $\mathsf{Support}_i(\mathbf{s})$ for player $i \in \mathcal{N}$. Also, we define $\mathsf{Paths}_v(\mathbf{s}) = \{p \in \mathsf{Paths}(G) : p \in \mathsf{Support}_{pp}(\mathbf{s}) \text{ and } v \in \mathsf{Vertices}_G(p)\}$. For a vertex $v \in V$, the probability the path player pp chooses a path that contains the vertex v is denoted $P_{\mathbf{s}}(\mathsf{Hit}(v))$. Clearly, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{p \in \mathsf{Paths}(v)} s_{pp}(p)$. For a vertex $v \in V$, we define $\mathsf{VP}_v(\mathbf{s})$ as in $\Pi_{\mathsf{E}}(G)$. So, for a path $p \in \mathsf{P}_G(\mathbf{s})$, we denote $\mathsf{VP}_p(\mathbf{s}) = \sum_{v \in \mathsf{Vertices}_G(p)} \sum_{i \in \mathcal{N}_{vp}} s_i(v)$.

Similarly to $\Pi_{\mathsf{E}}(G)$, we define the ***Expected Individual Profit*** $\mathsf{IP}_i\mathbf{s}$, and the ***Conditional Expected Individual Profit*** $\mathsf{IP}_i((\mathbf{s}_{-i}, x))$, of player i, for a strategy $x \in S_i$. So, for any vertex player $vp_i \in \mathcal{N}_{vp}$, for any vertex $v \in V$, $\mathsf{IP}_i((\mathbf{s}_{-i}, v)) = 1 - P_{\mathbf{s}}(\mathsf{Hit}(v))$. For the path player, for any path $p \in \mathsf{Paths}(G)$, $\mathsf{IP}_{pp}((\mathbf{s}_{-pp}, p)) = \mathsf{VP}_p(\mathbf{s})$.

2.4 Social Cost and Price of Anarchy

We utilize the notion of ***social cost*** [5] for evaluating the system performance.

Definition 2.1 *For model* M, $\mathsf{M} = \{\mathsf{P}, \mathsf{E}\}$, *we define the* ***social cost*** *of configuration* $\mathbf{s}$ *on instance* $\Pi_{\mathsf{M}}(G)$, $\mathsf{SC}(\Pi_{\mathsf{M}}, \mathbf{s})$, *to be the (expected) sum of vertex players of* Π_{M} *arrested in* $\mathbf{s}$. *That is,* $\mathsf{SC}(\Pi_{\mathsf{M}}, \mathbf{s}) = \mathsf{IP}_p(\mathbf{s})$ *(*$p = \{pp, ep\}$*, when* $\mathsf{M} = \mathsf{P}$ *and* $\mathsf{M} = \mathsf{E}$*, respectively).*

Obviously, the system wishes to *maximize* the social cost.

Definition 2.2 *For model* M, $\mathsf{M} = \{\mathsf{P}, \mathsf{E}\}$, *the* ***price of anarchy,*** $\mathsf{r}(\mathsf{M})$ *is*

$$\mathsf{r}(\mathsf{M}) = \max_{\Pi_{\mathsf{M}}(G), \mathbf{s}} \frac{\max_{\mathbf{s}^* \in \mathcal{S}} \mathsf{SC}(\Pi_{\mathsf{M}}(G), \mathbf{s}^*)}{\mathsf{SC}(\Pi_{\mathsf{M}}(G), \mathbf{s})}$$

3 The Edge Model

We first study Nash equilibria in the Edge model.

3.1 r-Factor, Regular, Perfect Matching Graphs

Theorem 3.1 *For a positive integer* r, *an* r*-factorizable graph* G *admits a Nash equilibrium* $\mathbf{s}$ *that can be computed in time* $O(T(G))$, *where* $O(T(G))$ *is the time needed for the computation of an* r*-factor of* G. *Also,* $\mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) = \frac{2\nu}{|V|}$.

Proof. From G first compute in polynomial time $O(T(G))$ an r-regular factor of G, $G_r(V, E_r)$. Then construct the following configuration $\mathbf{s}$ on $\Pi_{\mathsf{E}}(G)$:

For each vertex player $i \in \mathcal{N}_{vp}$, and each vertex $v \in V$, set $s_i(v) := \frac{1}{|V|}$.
For each edge $e \in E_r$, set $s_{ep}(e) := \frac{1}{|E_r|}$ and $s_{ep}(e) := 0$ for each edge $e \in E \backslash E_r$.

Note first that $\mathbf{s}$ is constructed in constant time. We now show that $\mathbf{s}$ is a Nash equilibrium. For any vertex $v \in V (= \mathsf{Support}_{vp}(\mathbf{s}))$,

$$
\begin{aligned}
& P_{\mathbf{s}}(\mathsf{Hit}(v)) \\
= & \sum_{e \in \mathsf{Edges}_v(\mathbf{s})} s_{ep}(e) \\
= & \frac{|\mathsf{Edges}_v(\mathbf{s})|}{|E_r|} \quad \text{(since } s_{ep} = \frac{1}{|E_r|}\text{, for each } e \in \mathsf{Support}_{ep}(\mathbf{s})\text{, by construction)} \\
= & \frac{r}{\frac{r \cdot |V|}{2}} \quad \text{(since } \Delta_{G(\mathsf{Support}_{ep}(\mathbf{s}))}(v) = r \text{ and } |E_r| = \frac{r \cdot |V|}{2}\text{)} \\
= & \frac{2}{|V|}.
\end{aligned}
$$

It follows that $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \min_{v' \in V} P_{\mathbf{s}}(\mathsf{Hit}(v'))$, for any vertex $v \in \mathsf{Support}_{vp}(\mathbf{s})$. Thus, $\mathbf{s}$ satisfies Condition (1) in the characterization of Nash equilibria (Theorem 2.1). For condition (2), consider any edge $e = (u, v) \in E$. Then,

$$
\begin{aligned}
& \mathsf{VP}_e(\mathbf{s}) \\
= & \mathsf{VP}_u(\mathbf{s}) + \mathsf{VP}_v(\mathbf{s}) \\
= & \sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v)) \\
= & 2 \cdot \nu \cdot \frac{1}{|V|}, \quad \text{(since } \forall\, v \in V,\ v \in \mathsf{Support}_{vp}(\mathbf{s}) \text{ and } s_i(v) = \frac{1}{|V|}\text{, by construction)}
\end{aligned}
$$

It follows that $\mathsf{VP}_e(\mathbf{s}) = \max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s})$, for each edge $e \in \mathsf{Support}_{ep}(\mathbf{s})$. Thus, $\mathbf{s}$ satisfies Condition (2) in the characterization of Nash equilibria (Theorem 2.1), which proves that it is a Nash equilibrium.

We finally compute the social cost of $\mathbf{s}$:

$$
\begin{aligned}
& \mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) \\
= & \mathsf{IP}_{ep}(\mathbf{s}) \\
= & \mathsf{VP}_e(\mathbf{s}), \quad \text{(for any edge } e \in \mathsf{Support}_{ep}(\mathbf{s})\text{, since } \mathbf{s} \text{ is a Nash equilibrium)} \\
= & \frac{2\nu}{|V|},
\end{aligned}
$$

as required. ■

Corollary 3.2 *For a positive integer r, a polynomially computable r-factorizable graph G has a polynomially computable Nash equilibrium.*

Corollary 3.3 *A regular graph has a polynomially computable Nash equilibrium.*

Observe that the class of r-factorizable graphs is a subclass for $(r-1)$-factorizable graphs. So, we can restrict ourselves to the easiest case of 1-factorizable graphs (Perfect Matching graphs). Such a graph can be recognized in polynomial time $O\left(\sqrt{|V|}|E|\cdot\ \log_{|V|}\frac{|V|^2}{|E|}\right)$ [3]; and a Perfect Matching of it can be can be computed in the same time. Thus, Theorem 3.1 implies that,

Corollary 3.4 *If G is a Perfect Matching graph, then it admits a Nash equilibrium that can be computed in time* $O\left(\sqrt{|V|}|E|\cdot\log_{|V|}\frac{|V|^2}{|E|}\right)$.

3.2 Trees

In this section we consider trees. We present an efficient algorithm to compute a mixed Nash equilibrium in a tree. The algorithm, called TreesNE, is presented in pseudocode in Figure 3.2.

Algorithm TreesNE
INPUT: A rooted tree $T = (V, E)$
OUTPUT: A mixed Nash equilibrium $\mathbf{s}$ for T.

(1) Set $T' := T$ and $VC := \emptyset$.

(2) Repeat until $T' == \emptyset$

(2/a) Set $VC := VC \cup \mathsf{leaves}(T')$.

(2/b) For each vertex $v \in \mathsf{leaves}(T')$,

If $\mathsf{parent}_{T'}(v) \neq \emptyset$, then $EC := EC \cup \{(v, \mathsf{parent}_{T'}(v)))\}$
else $EC := EC \cup \{(v, u)\}$, for any $u \in \mathsf{children}_T(v)$.

(2/c) Set $T' := T' \backslash \{\mathsf{leaves}(T'), \mathsf{parents}(\mathsf{leaves}(T'))\}$.

(3) Set $\mathsf{Support}_{ep}(\mathbf{s}) := EC$ and $\mathsf{Support}_{vp}(\mathbf{s}) := VC$.

(3) For each vertex player $i \in \mathcal{N}_{vp}$, and each vertex $v \in VC$, set $s_i(v) = \frac{1}{|VC|}$. For each edge $e \in EC$, set $s_{ep}(e) = \frac{1}{|EC|}$.

Theorem 3.5 *Algorithm* TreesNE *computes a mixed Nash equilibrium of a tree graph $T = (V, E)$ in linear time $O(|V|)$.*

Proof. We prove the claim via a sequence of Lemmas.

Lemma 3.6 *Set VC is an Independent Set.*

Proof. Set VC is constructed during Step (2) of the algorithm. We prove that the set is an Independent Set by induction to the number of iterations of the step. Consider an iteration r and denote as T'_r and VC_r the current graph T' and vertex set VC, respectively. In the first iteration of the step, set VC is extended with the leaves of tree T'_r, $\mathsf{leaves}(T'_r)$. Since for the first iteration, $\mathsf{leaves}(T') = \mathsf{leaves}(T)$, it follows that initially, set VC_r is an Independent Set.

Assume, by induction, that VC_r is an Independent Set at the beginning of iteration r. We prove that this remains true after the extension of the set at the end of the iteration. Consider a vertex $v \in \mathsf{leaves}(T'_r)$. Since $r > 1$, vertex v is an inner vertex of the graph T. Moreover, by Step (2/c), in each expansion of set VC, we exclude from the set the parents of the currently inserted vertices. It follows that vertex set $\mathsf{children}_T(v)$ has been excluded from the set VC_r, in previous iterations of the step. As it concerns the parent of v in T, we argue that $\mathsf{parent}_T(v)$ is not contained in VC_r. This is true because, by Step (2/a), in each iteration, VC_r is expanded only with the leaves of T_r. It follows that $\mathsf{parent}_T(v)$ can not be contained in VC_r. Thus, extending set VC with set $\mathsf{leaves}(T'_r)$, in current iteration, guarantees that the set is still an Independent Set, as required. ■

Lemma 3.7 *Set EC is an Edge Cover.*

Proof. Note that each vertex $v \in V$ is considered in Step (2) of the algorithm, either as a *(i)* leaf of T' or *(ii)* a parent of such a leaf. In both cases, an edge (v, u) is added to EC (in Step (2/b)), where u is the parent of vertex v, for case *(i)* or u is one of its children, for case *(ii)*. Thus, in any case, vertex v is covered by EC so that EC is an Edge Cover. ■

Lemma 3.8 *Set VC is a Vertex Cover of $T(EC)$.*

Proof. For each edge (v, u) added to EC (at Step (2/b) of the algorithm), vertex $v \in \mathsf{leaves}(T')$ was added to VC (at Step (2/a) of the same iteration of the algorithm). The claim follows. ■

Lemma 3.9 *Profile* $\mathbf{s}$ *is an Independent Covering profile.*

Proof. We first show that $\mathbf{s}$ is a Covering profile. By Lemma 3.7, EC is an Edge Cover. By Step (3) of the algorithm, $\mathsf{Support}_{ep}(\mathbf{s}) = EC$. It follows that $\mathsf{Support}_{ep}(\mathbf{s})$ is an Edge Cover, as required by Condition (1) of a Covering profile. By Lemma 3.8, VC is a Vertex Cover of $T(EC)$. By Step (3) of the algorithm, $\mathsf{Support}_{ep}(\mathbf{s}) = EC$ and $\mathsf{Support}_{vp}(\mathbf{s}) = VC$. It follows that $\mathsf{Support}_{vp}(\mathbf{s})$ is a Vertex Cover of the graph $T(\mathsf{Support}_{ep}(\mathbf{s}))$, as required by Condition (2) of a Covering profile. Thus, $\mathbf{s}$ is a Covering profile.

We proceed to show that $\mathbf{s}$ is an Independent Covering profile. Note first that $\mathbf{s}$ is a uniform profile such that $s_i = s_j$, for any two vertex players $i, j \in \mathcal{N}_{vp}$. By Lemma 3.6, and since $\mathsf{Support}_{vp}(\mathbf{s}) = VC$, it follows that $\mathsf{Support}_{vp}(\mathbf{s})$ is an Independent Set. Thus, $\mathbf{s}$ satisfies additional Condition (1) of an Independent Covering profile.

We proceed to prove additional Condition (2) of an Independent Covering profile. Note that, by Step (2/b) of the algorithm, for each vertex v added in VC, we add an edge (v, u) in EC, where either $u = \mathsf{parent}_{T'}(v)$ or $u \in \mathsf{children}_T(v)$ (in the case where $\mathsf{parent}_T(v) = \emptyset$). By Condition (1) of a Covering profile, there exists at least one edge $(v, u) \in \mathsf{Support}_{ep}(\mathbf{s})$. We claim that there is exactly one such edge. Note that, vertex v is either a leave of T or an inner vertex of T. In the first case, edge (v, u) is the only incident to v edge in T; thus the claim follows. For the latter case, since $v \in \mathsf{leaves}(T')$, Step (2/c) implies that $\mathsf{children}_T(v)$ have been removed from T' in previous iterations of the algorithm. Thus, edge (v, u), such that $u \in \mathsf{children}_T(v)$ are not contained in $\mathsf{Support}_{ep}(\mathbf{s})$. Thus, edge (v, u), such that $u \in \mathsf{children}_T(v)$ are not contained in $\mathsf{Support}_{ep}(\mathbf{s})$. Since, T is a tree, it follows that edge $(v, u) \in \mathsf{Support}_{ep}(\mathbf{s})$ is the only one edge of $\mathsf{Support}_{ep}(\mathbf{s})$ incident to vertex $v \in \mathsf{Support}_{vp}(\mathbf{s})$. Additional Condition (2) of an Independent Covering profile follows, concluding the claim. ■

We now return in the proof of the Theorem. Lemma 3.9 implies that $\mathbf{s}$ is an Independent Covering profile. Thus, by Theorem 2.3, $\mathbf{s}$ is a Nash equilibrium, as required.

For the time complexity of the algorithm, Step (1) takes constant time. Step (2) iterates at most $O(|V|)$ times. In each iteration, Steps (2/a), (2/c) takes constant time. By Step (2/b), each vertex $v \in V$ is considered in exactly one iteration of Step (2). Moreover, Step (2/b) takes constant time. It follows that Step (2) is completed in linear time $O(|V|)$. Step (3) takes constant time. Finally, Step (4) takes $O(|V|)$ time. It follows that the algorithm is completed in linear time $O(|V|)$. ■

3.3 The Price of Anarchy

We first upper and lower bound the Social Cost of a Nash equilibrium.

Theorem 3.10 *For a Nash equilibrium* $\mathbf{s}$*,*

$$\frac{\nu}{\min\left\{|\mathsf{Support}_{ep}(\mathbf{s})|, |\mathsf{Support}_{vp}(\mathbf{s})|\right\}} \leq \mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) \leq \frac{\nu \cdot \Delta_{G(\mathsf{Support}_{ep}(\mathbf{s}))}}{|\mathsf{Support}_{ep}(\mathbf{s})|}.$$

Proof. We first show the lower bound.

$$\begin{aligned}
& \max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s}) \\
= & \frac{1}{|\mathsf{Support}_{ep}(\mathbf{s})|} \cdot \textstyle\sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} \mathsf{VP}_e(\mathbf{s}) \quad \text{(by Condition (2) of Theorem 2.1).} \\
= & \frac{1}{|\mathsf{Support}_{ep}(\mathbf{s})|} \cdot \textstyle\sum_{v \in \mathsf{Vertices}_G(\mathsf{Support}_{ep}(\mathbf{s}))} |\mathsf{Edges}_v(\mathbf{s})| \cdot \mathsf{VP}_v(\mathbf{s}) \\
= & \frac{1}{|\mathsf{Support}_{ep}(\mathbf{s})|} \cdot \textstyle\sum_{v \in V} |\mathsf{Edges}_v(\mathbf{s})| \cdot \mathsf{VP}_v(\mathbf{s}),
\end{aligned}$$

since profile $\mathbf{s}$ is a Covering profile (by Proposition 2.2), it satisfies Condition (1) of Covering profile, i.e. $\mathsf{Support}_{ep}(\mathbf{s})$ is an Edge Cover of G. So,

$$\begin{aligned}
& \max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s}) \\
\geq & \frac{1}{|\mathsf{Support}_{ep}(\mathbf{s})|} \cdot \textstyle\sum_{v \in V} \mathsf{VP}_v(\mathbf{s}) \\
= & \frac{\nu}{|\mathsf{Support}_{ep}(\mathbf{s})|}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}) \\
= & \mathsf{IP}_{ep}(\mathbf{s}) \\
= & \mathsf{VP}_e(\mathbf{s}) \quad \text{(for an edge } e \in \mathsf{Support}_{ep}(\mathbf{s}) \text{, since } \mathbf{s} \text{ is a Nash equilibrium)} \\
= & \max\nolimits_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s}) \quad \text{(by Condition (2) of Theorem 2.1)} \\
\geq & \frac{\nu}{|\mathsf{Support}_{ep}(\mathbf{s})|}. \qquad (1)
\end{aligned}$$

Also, we argue that since $\mathbf{s}$ is a Nash equilibrium, there exists a vertex $v' \in \mathsf{Support}_{vp}(\mathbf{s})$ such that $\mathsf{VP}_v(\mathbf{s}) \geq \frac{\nu}{|\mathsf{Support}_{vp}(\mathbf{s})|}$. Assume in contrary that, $\mathsf{VP}_v(\mathbf{s}) < \frac{\nu}{|\mathsf{Support}_{vp}(\mathbf{s})|}$, for each vertex $v \in \mathsf{Support}_{vp}(\mathbf{s})$. Then,

$$\begin{aligned}
& \sum_{v \in V} \mathsf{VP}_v(\mathbf{s}) \\
= & \textstyle\sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})} \mathsf{VP}_v(\mathbf{s}) \\
< & |\mathsf{Support}_{vp}(\mathbf{s})| \cdot \frac{\nu}{|\mathsf{Support}_{vp}(\mathbf{s})|} \\
< & \nu,
\end{aligned}$$

a contradiction since, $\sum_{v \in V} \mathsf{VP}_v(\mathbf{s}) = \nu$ (since $\mathbf{s}$ is a profile).

It follows that there exists a vertex $v' \in \mathsf{Support}_{vp}(\mathbf{s})$ such that $\mathsf{VP}_{v'}(\mathbf{s}) \geq \frac{\nu}{|\mathsf{Support}_{vp}(\mathbf{s})|}$.

Thus,

$$\begin{aligned}
&\mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) \\
= &\quad \mathsf{IP}_{ep}(\mathbf{s}) \\
= &\quad \mathsf{VP}_{e'}(\mathbf{s}) &&\text{(for an edge } e' = (v', u) \in \mathsf{Support}_{ep}(\mathbf{s})\text{, since } \mathbf{s} \text{ is a Nash equilibrium)} \\
= &\quad \mathsf{VP}_{v'}(\mathbf{s}) + \mathsf{VP}_{u}(\mathbf{s}) \\
\geq &\quad \mathsf{VP}_{v'}(\mathbf{s}) \\
\geq &\quad \frac{\nu}{|\mathsf{Support}_{vp}(\mathbf{s})|} &&\text{(by definition of vertex } v'\text{),}
\end{aligned}$$

as required.

It folllows that,

$$\mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) \quad \geq \quad \frac{\nu}{\min\left\{|\mathsf{Support}_{ep}(\mathbf{s})|, |\mathsf{Support}_{vp}(\mathbf{s})|\right\}}.$$

We now prove the upper bound.

$$\begin{aligned}
&\sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} \mathsf{VP}_{e}(\mathbf{s}) \\
= &\quad \textstyle\sum_{v \in \mathsf{Vertices}_G(\mathsf{Support}_{ep}(\mathbf{s}))} |\mathsf{Edges}_{v}(\mathbf{s})| \cdot \mathsf{VP}_{v}(\mathbf{s}) \\
= &\quad \textstyle\sum_{v \in V} |\mathsf{Edges}_{v}(\mathbf{s})| \cdot \mathsf{VP}_{v}(\mathbf{s}) \\
\leq &\quad \textstyle\sum_{v \in V} \Delta(G(\mathsf{Support}_{ep}(\mathbf{s}))) \cdot \mathsf{VP}_{v}(\mathbf{s}) \\
= &\quad \Delta(G(\mathsf{Support}_{ep}(\mathbf{s}))) \cdot \textstyle\sum_{v \in V} \mathsf{VP}_{v}(\mathbf{s}) \\
\leq &\quad \Delta(G(\mathsf{Support}_{ep}(\mathbf{s}))) \cdot \nu &&(\mathbf{s} \text{ is a profile})
\end{aligned}$$

So,

$$\begin{aligned}
&\max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s}) \\
= &\quad \frac{1}{|\mathsf{Support}_{ep}(\mathbf{s})|} \cdot \textstyle\sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} \mathsf{VP}_{e}(\mathbf{s}) &&\text{(by Condition (2) of Theorem 2.1)} \\
= &\quad \frac{\Delta(G(\mathsf{Support}_{ep}(\mathbf{s}))) \cdot \nu}{|\mathsf{Support}_{ep}(\mathbf{s})|}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}) \\
= &\quad \mathsf{IP}_{ep}(\mathbf{s}) \\
= &\quad \mathsf{VP}_{e}(\mathbf{s}) &&\text{(for an edge } e \in \mathsf{Support}_{ep}(\mathbf{s})\text{, since } \mathbf{s} \text{ is a Nash equilibrium)} \\
= &\quad \max_{e' \in E} \mathsf{VP}_{e'}(\mathbf{s}) &&\text{(by Condition (2) of Theorem 2.1)} \\
\leq &\quad \frac{\Delta(G(\mathsf{Support}_{ep}(\mathbf{s}))) \cdot \nu}{|\mathsf{Support}_{ep}(\mathbf{s})|}.
\end{aligned}$$

The proof is now complete. ■

We provide an estimation on the payoffs of the vertex players in any Nash equilibrium.

Lemma 3.11 *In a mixed Nash equilibrium* $\mathbf{s}$*,*

$$1 - \frac{2}{|\mathsf{Support}_{vp}(\mathbf{s})|} \leq \mathsf{IP}_i(\mathbf{s}) \leq 1 - \frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|}.$$

Proof. For any vertex player $i \in \mathcal{N}_{vp}$,

$$\begin{aligned}
&\mathsf{IP}_i(\mathbf{s}) \\
= &\frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|} \cdot \sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})} \mathsf{IP}_i(\mathbf{s}) \\
= &\frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|} \cdot \sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})} \left(1 - \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} s_{ep}(e)\right) \\
= &\frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|} \cdot \sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})} (1) - \sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})} \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} s_{ep}(e)
\end{aligned}$$

Since $\mathbf{s}$ is a Covering profile (Proposition 2.2), $\mathsf{Support}_{vp}(\mathbf{s})$ is a vertex cover of $\mathsf{Support}_{ep}(\mathbf{s})$. Therefore, any edge $e \in \mathsf{Support}_{ep}(\mathbf{s})$ appears either at most twice in the the second right sum $\sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})}$, once for each of its endpoints. So,

$$\begin{aligned}
&\mathsf{IP}_i(\mathbf{s}) \\
\geq &\frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|} \cdot \left(|\mathsf{Support}_{vp}(\mathbf{s})| - 2 \sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} s_{ep}(e)\right) \quad (\text{since } \sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} s_{ep}(e) = 1) \\
= &1 - \frac{2}{|\mathsf{Support}_{vp}(\mathbf{s})|},
\end{aligned}$$

as required.

Moreover, any edge $e \in \mathsf{Support}_{ep}(\mathbf{s})$ appears either at least once in the the second right sum $\sum_{v \in \mathsf{Support}_{vp}(\mathbf{s})}$, for one of the two end points of edge e. So,

$$\begin{aligned}
&\mathsf{IP}_i(\mathbf{s}) \\
\leq &\frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|} \cdot \left(|\mathsf{Support}_{vp}(\mathbf{s})| - 1 \sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} s_{ep}(e)\right) \quad (\text{since } \sum_{e \in \mathsf{Support}_{ep}(\mathbf{s})} s_{ep}(e) = 1) \\
= &1 - \frac{1}{|\mathsf{Support}_{vp}(\mathbf{s})|},
\end{aligned}$$

as required. The claims follows. ■

Theorem 3.12 *For the Edge model,* $\frac{|V|}{2} \leq \mathsf{r}(\mathsf{E}) \leq |V|$*.*

Proof. We first prove the upper bound. Obviously, $\max_{\mathbf{s}^* \in \mathcal{S}} \mathsf{SC}(\Pi_{\mathsf{E}}(G), \mathbf{s}^*) \leq \nu$. Now consider the following profile $\mathbf{s}'$: Set each vertex player $i \in \mathcal{N}_{vp}$, $s'_i = v$, for some vertex $v \in V$ and set

$s'_{ep} = (v, u)$, for some $(v, u) \in E$ such that $u \in \mathsf{Neigh}_G(v)$. Then, $\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}^*) = \mathsf{IP}_{ep}(\mathbf{s}') = \nu$. It follows that, $\max_{\mathbf{s}^* \in \mathcal{S}} \mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}^*) = \nu$. So,

$$
\begin{aligned}
&\mathsf{r}(\mathsf{E}) \\
= \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \frac{\max_{\mathbf{s}^* \in \mathcal{S}} \mathsf{SC}(\Pi_\mathsf{M}(G), \mathbf{s}^*)}{\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s})} \\
= \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \frac{\nu}{\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s})} && \text{(since } \mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}') = \nu\text{)} \\
\leq \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \frac{\nu}{\frac{\nu}{\min\left\{|\mathsf{Support}_{ep}(\mathbf{s})|, |\mathsf{Support}_{vp}(\mathbf{s})|\right\}}} && \text{(by Theorem 3.10)} \\
\leq \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \{\min\{|\mathsf{Support}_{ep}(\mathbf{s})|, |V(\mathsf{Support}_{vp}(\mathbf{s}))|\}\} \\
\leq \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \{|V(\mathsf{Support}_{vp}(\mathbf{s}))|\}
\end{aligned}
$$

The last inequality is is true because $\max\{\min\{f_1, f_2\}\} \leq \max\{f_1\}$, where f_1, f_2 are any two functions. So,

$$
\begin{aligned}
\mathsf{r}(\mathsf{E}) \leq \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \{|V(\mathsf{Support}_{vp}(\mathbf{s}))|\} \\
\leq \quad & |V| && \text{(since } |\mathsf{Support}_{vp}(\mathbf{s})| \leq |V|\text{)}.
\end{aligned}
$$

We proceed to show the lower bound. Note that Theorem 3.3 implies that there exists an instance $\Pi_\mathsf{E}(G)$ and a profile $\mathbf{s}'$ such that $\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}') = \frac{2\nu}{|V|}$. Thus,

$$
\begin{aligned}
&\mathsf{r}(\mathsf{E}) \\
= \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \frac{\max_{\mathbf{s}^* \in \mathcal{S}} \mathsf{SC}(\Pi_\mathsf{M}(G), \mathbf{s}^*)}{\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s})} \\
= \quad & \max_{\Pi_\mathsf{E}(G),\mathbf{s}} \frac{\nu}{\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s})} && \text{(since } \mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}^*) = \nu\text{)} \\
\geq \quad & \frac{\nu}{\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}')} && \text{(by definition of profile } \mathbf{s}'\text{)} \\
\geq \quad & \frac{\nu}{\frac{2\nu}{|V|}} && \text{(}\mathsf{SC}(\Pi_\mathsf{E}(G), \mathbf{s}') = \frac{2\nu}{|V|}\text{, by Theorem 3.3)} \\
= \quad & \frac{|V|}{2},
\end{aligned}
$$

as required. ■

4 The Path Model

For the Path model, we characterize pure Nash Equilibria. We first prove:

Proposition 4.1 *If $\Pi_\mathsf{P}(G)$ admits a pure Nash equilibrium $\mathbf{s}$ then the graph G is Hamiltonian.*

Proof. Consider the path $w \in \mathsf{Paths}(G)$ such that $s_{pp}(w) = 1$. We prove that w is a Hamiltonian path so that G is Hamiltonian. Assume in contrary that G does *not* contain a Hamiltonian path. Then, $\mathsf{Vertices}_G(w) \subset V$. $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{w \in \mathsf{Paths}_v(\mathbf{s})} s_{pp}(w) = 1$. On the other hand, for any vertex $v \in V \backslash \{\mathsf{Vertices}_G(w)\}$, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{w \in \mathsf{Paths}_v(\mathbf{s})} s_{pp}(w) = 0$.

Since, $\mathsf{IP}_i((\mathbf{s}_{-i}, v)) = 1 - P_{\mathbf{s}}(\mathsf{Hit}(v))$, it follows that $\mathsf{IP}_i((\mathbf{s}_{-i}, v)) > \mathsf{IP}_i((\mathbf{s}_{-i}, u))$, for any two vertices v, u, such that $v \in V \backslash \{\mathsf{Vertices}_G(w)\}$ and $u \in \mathsf{Vertices}_G(w)$. Since $\mathbf{s}$ is a local maximizer for the Expected Individual Profit of the player vp_i and a pure profile, it follows that the player chooses some such v with probability 1 while he chooses any such u with probability zero. It follows that,

$$\begin{aligned} \mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w)) &= \mathsf{VP}_w(\mathbf{s}) \\ &= \sum_{v \in \mathsf{Vertices}_G(w)} \sum_{i \in \mathcal{N}_{vp}} s_i(v) \\ &= 0. \end{aligned}$$

Consider now an alternative path $w' \in \mathsf{Paths}(G)$, $w' \neq w$, such that $v \in \mathsf{Vertices}_G(w')$ and $v \in V \backslash \{\mathsf{Vertices}_G(w)\}$. (Note that for vertex v, $s_i(v) = 1$). Then,

$$\begin{aligned} \mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w')) &= \mathsf{VP}_{w'}(\mathbf{s}) \\ &= \sum_{v \in \mathsf{Vertices}_G(w')} \sum_{i \in \mathcal{N}_{vp}} s_i(v) \\ &> 0. \end{aligned}$$

Since $\mathbf{s}$ is a Nash equilibrium, for each strategy $w \in S_{pp}$ such that $s_{pp}(w) > 0$, all Conditional Expected Individual Profits $\mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w))$ are the same and no less than any Conditional Expected Individual Profit $\mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w'))$ with $s_{pp}(w') = 0$. Thus,

$$\begin{aligned} \mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w)) \geq & \ \mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w')) \\ > & \ 0, \end{aligned}$$

a contradiction, since $\mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w)) = 0$. It follows that G is Hamiltonian, as required. ■

Moreover, we prove:

Theorem 4.2 *If the graph G is Hamiltonian then $\Pi_{\mathsf{P}}(G)$ admits a pure Nash equilibrium.*

Proof. Assume that G contains a Hamiltonian path w. Consider a pure profile $\mathbf{s}$ such that $s_{pp}(w) = 1$. Then, for the path player pp,

$$\begin{aligned} & \mathsf{IP}_{pp}((\mathbf{s}_{-pp}, w)) \\ = & \quad \mathsf{VP}_w(\mathbf{s}) \\ = & \quad \textstyle\sum_{v \in \mathsf{Vertices}_G(w)} \sum_{i \in \mathcal{N}_{vp}} s_i(v) \\ = & \quad \nu \qquad \text{(since } w \text{ is a Hamiltonian path, by construction).} \end{aligned}$$

Thus, the Conditional Expected Individual Profit $\mathsf{IP}_{pp}(\mathbf{s}_{-pp}, w)$ of the path player is $\mathsf{IP}_{pp}(\mathbf{s}_{-pp}, w) \geq \mathsf{IP}_{pp}(\mathbf{s}_{-pp}, w')$, for any alternative strategy $w' \in \mathsf{Paths}(G)$. It follows that $\mathbf{s}$ is a local maximizer for its Expected Individual Profit.

As, it concerns the vertex players, note that for any vertex $v \in V$, it holds that $v \in \mathsf{Vertices}_G(w)$. Thus, for any vertex player $vp_i \in \mathcal{N}_{vp}$, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{w \in \mathsf{Paths}_v(\mathbf{s})} s_{pp}(w) = 1$, for any vertex $v \in V$. Since the Conditional Expected Individual Profit $\mathsf{IP}_i((\mathbf{s}_{-i}, v)) = 1 - P_{\mathbf{s}}(\mathsf{Hit}(v))$, it follows that $\mathsf{IP}_i((\mathbf{s}_{-i}, v)) = 0$, for any vertex $v \in V$. It follows that $\mathbf{s}$ is a local maximizer for its Expected Individual Profit. Since $\mathbf{s}$ is a local maximizer for the Expected Individual Profits of all players, it is a pure Nash equilibrium, as required. ■

Propositions 4.1 and 4.2 together imply that:

Theorem 4.3 *$\Pi_{\mathsf{P}}(G)$ admits a pure Nash equilibrium if and only if the graph G is Hamiltonian.*

Theorem 4.3 immediately implies:

Corollary 4.4 *The problem of deciding whether there exists a pure NE for any $\Pi_{\mathsf{P}}(G)$ is $\mathcal{NP}$-complete.*

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