Hardness Results and Efficient Approximations for Frequency Assignment Problems: Radio Labelling and Radio Coloring

D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou, and P.G. Spirakis *

Computer Technology Institute (CTI) and Patras University, Greece
Riga Fereou 61, 26221 Patras, Greece
emails: {fotakis,nikole,viki,spirakis}@cti.gr

Abstract. The *Frequency Assignment Problem* (FAP) in radio networks is the problem of assigning frequencies to transmitters, exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modeled by variations of graph coloring. In this work we first survey the variations of the FAP and then we study two (similar but still different) frequency assignment problems: *radio labelling* and *radiocoloring*.
For radio labelling, we prove that the problem is $\mathcal{NP}$-complete for general graphs and we provide efficient $\mathcal{NC}$ approximation algorithms. We also give a polynomial time algorithm computing an optimal radio labelling for planar graphs thus, showing that radio labelling is in $\mathcal{P}$ for planar graphs. On the other hand, we prove that radiocoloring remains $\mathcal{NP}$-complete even for planar graphs and we provide an efficient 2-ratio approximation algorithm. We also present a fully polynomial randomized approximation scheme for computing the number of different radiocolorings of planar graphs.
Keywords: mobile computing, radio networks, coloring, computational complexity, approximations, planar graphs, rapid mixing, coupling.

1 Introduction: Frequency Assignment Problems (FAP)

The recent remarkable explosion of wireless and mobile networks together with the limited frequency spectrum necessitate the design of algorithms for the efficient use of the available frequencies.

The Frequency Assignment Problem, henceforth abbreviated as FAP, is the problem of assigning frequencies to the stations of a network, so that interference between nearby stations is avoided or minimized while the frequency reusability is exploited. For a network G of stations, a feasible assignment of frequencies f assigns to each transmitter u a frequency $f(u)$ that satisfies certain frequency and distance constraints. An efficient assignment tries to minimize a cost function depending on the objective of the assignment.

Two typical objectives of FAP are given below:

- minimize the range of frequencies (bandwidth) used, called *span*. Real networks reserve bandwidth (range of frequencies) rather than distinct frequencies. The objective of an efficient assignment in such networks is to minimize the bandwidth (span) used. The span of an assignment is the difference between the largest and the smallest frequency used in the assignment.
- minimize the number of distinct frequencies used in the assignment, called *order*. There are cases where the objective is to minimize the number of distinct frequencies used so that unused frequencies can be available for other use, thus increasing the system's performance.

Moreover, the frequency assignment objectives may be related to the level of system interference:

- minimize the bandwidth or the number of distinct frequencies used subject to an acceptable level of interference,

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– minimize the system interference subject to an acceptable given frequency allocation.

Finally, a frequency assignment may have two objectives, a primary and a secondary [42, 39]:

– In networks where the reserved bandwidth (span) is considered as the primary objective of an assignment, it is usually desirable to use as less distinct frequencies of the used bandwidth (span) as possible, so that the unused frequencies to be available for other use by the system. In such networks the primary objective is to minimize the span of frequencies used in the assignment while the secondary objective is to minimize the order.
– On the other hand, there are cases where the primary objective of an assignment is to minimize the distinct number of frequencies used and a secondary goal to minimize the span, since it is desirable not to reserve unnecessary bandwidth. In these networks, the primary objective is to minimize the order of the frequency assignment, while the secondary objective is to minimize the span.

In Frequency Assignment Problems, there are several types of interference constraints [42, 73]:

– Frequency-distance constraints. Such constraints relate the distance between transmitters' physical locations and the difference between their respective frequencies. As an example, in a Cellular Network Base Station modeled by a graph G(V,E), the frequency $f(u)$ assigned to a station u, should be such so that if $d(u, v) \leq D$ then $f(u) \neq f(v)$, for each $v \in V$, where $d(u, v)$ is the distance between the two stations and D the minimum frequency reuse distance. More specifically, the frequency-distance constraints may be expressed as a pair $FD(k) = [T(k), D(k)]$ for $k = 1, \ldots, K$, where $T(k)$ are the frequency constraints applied in vertices of distance $D(k)$ or smaller.
– Frequency constraints. In this case the interference constraints may be a function only of frequency. Constraints of this type are met in networks where the distances between transmitters are unknown or uncontrollable.
– Co-channel constraints. Such interference constraints force two typically nearby transmitters not to be assigned the same channel.
– Adjacent channel constraints: Such constraints force a pair of typically nearby transmitters to be assigned channels of frequency distance at least one apart.

1.1 Graph Formulations of Frequency Assignment Problems

Frequency assignment problems are usually modeled in an abstract way (which enables the use of mathematical and algorithmic techniques in order to derive efficient solutions), by the notion of the interference graph. Each vertex of an interference graph represents a transmitter, and each edge represents an interference constraint between nearby transmitters. Under the assumption that a pair of adjacent transmitters should be assigned different frequencies, the problem of frequency assignment becomes equivalent to the problem of coloring the interference graph with the minimum number of colors. In graph coloring, we assign colors to the vertices of a graph, so that no two adjacent vertices get the same color. In this way, we get the following very basic graph-theoretic formulation of a frequency assignment problems:

Definition 1. Vertex coloring: *The simplest variation of FAP modeling only co-channel interference between neighbor stations, is equivalent to the graph coloring problem.*

Although being very intuitive and fundamental, graph coloring provides only a very limited model for the variety of frequency assignment problems. This fact gives rise to many different and more refined graph theoretic formulations.

A more general and comprehensive definition of FAP considering frequency-distance constraints is the following:

Definition 2. **F*D-FAP** *[29]: This formulation of FAP models frequency-distance constraints. The FAP problem is defined by the pairs* $D(k), T(k)$, $i = 1, \ldots, K$, *where* $T(k) \subset \mathbb{N}$, $D(k) \in \mathbb{N}$, *and* $0 = T(0) \subset T(1) \subset \ldots \subset T(K)$, $D(0) > D(1) \ldots > D(K) > 0$. *Then*

$$f(u) - f(v) \notin T(k), \forall u, v \in V \; iff \; d(u,v) \leq D(k)$$

A different coloring variation modeling constraints appearing in UHF-TV transmitters is the following:

Definition 3. **T-coloring** *[20]: This variation of FAP considers only specific frequency constraints. The assigned color of each vertex is chosen so that the frequency distance between colors of neighbor vertices is not contained in the forbidden set of frequencies* T. *A T-coloring of a graph* $G(V, E)$ *is a mapping* $f : V \to N$ *so that :*

$$(u, v) \;\in\; E(G) \;\Leftrightarrow |f(u) - f(v)| \notin T, \;\forall u, v \in V$$

A variation of coloring and T-coloring, in which the channels must be consecutive is the following:

Definition 4. **No-hale coloring** *[87]: The coloring of a graph such that the colors assigned are consecutive integers is called* no-hale coloring. *The T-coloring of* G *such that the colors assigned are consecutive integers is called* T-set no hale coloring.

It can be easily seen that the minimum number of colors needed for T-coloring, $\chi_T(G)$ is $\chi_T(G) = \chi(G)$, where $\chi(G)$ is the vertex chromatic number of G. However, the minimum span of any T-coloring, called $sp_T(G)$, is different from the minimum span of the usual coloring of the graph G.

In some networks (e.g. in cellular mobile networks) the use of some frequencies is forbidden for several base stations. This problem can be modeled by list coloring. In this coloring variation there is a list $L(u)$ associated with every vertex $u \in V$. The question is whether there is a proper coloring of the graph assigning to every vertex u a color from its own list.

Definition 5. **List Coloring** *[26]: Assign to each vertex a proper color from its list of available colors so that neighboring vertices get different colors.*

A very important notion for the theory of list-coloring is the *choice chromatic number*. It is the smallest number k such that the graph G can be properly colored with colors from the lists $L(u)$ provided that each $L(u)$ has at least k colors. If the choice number of a graph G is not larger than k then G is said to be $k-$*choosable*. Probably the most famous problem in that content is the 5-choosability of planar graphs conjectured by Vizing in 1975 and remained open until Thomassen proved it in 1993 [92].

Both constraints modeled by T-coloring and List Coloring may appear at the same time in the requested assignment of some networks:

Definition 6. **L-list T-coloring** *[91]: Given any T-set and list sets* $L(i)$, $i = 1, \ldots |V|$, *an L-list T-coloring of graph* $G(V, E)$ *is an assignment* f *so that*

$$V \to \mathbb{N} \text{ s.t.} (\mathrm{v_i}, \mathrm{v_j}) \in \mathrm{E} \Leftrightarrow |\mathrm{f}(\mathrm{v_i}) - \mathrm{f}(\mathrm{v_j})| \notin \mathrm{T} \text{ and } \mathrm{f}(\mathrm{v_i}) \in \mathrm{L(i)}, \;\forall \mathrm{v_i} \in \mathrm{V}$$

There follows a variation motivated by the frequency assignment problem in mobile radio stations where each station is required to be assigned $r(u)$ frequencies, where $r(u)$ is a small positive number.

Definition 7. **Set coloring** *[73]: The Set Coloring is a generalization of vertex coloring, consisting of assigning to every vertex* $u \in V$ *a set of colors so that neighbor vertices get disjoint sets of colors.*

We next provide three more variations of coloring, which are related to the problems of radio labelling and radiocoloring that we study in this work.

Definition 8. k-coloring *[29]: In this simplified version of frequency-distance constraints, the colors* $f(u), f(v)$ *assigned to any two vertices* u, v *satisfy* $f(u) - f(v) = \chi$ *for some* $\chi = 0, \ldots, k$ *only if* $d(u, v) \geq k - \chi + 1$.

Definition 9. distance-k-coloring *[67]: The colors* $f(u), f(v)$ *assigned to any two vertices* u, v *of distance at most* k *should satisfy* $f(u) \neq f(v)$.

Definition 10. Hidden Terminal Interference Avoidance Problem (HTIA) *[8]: color the vertices of a graph so that vertices* at distance exactly 2 *get different colors.*

In this work, we shall henceforth concentrate on the problems of radio labelling and radio coloring.

Definition 11. Radio Labelling *[30, 48]: The colors* $f(u), f(v)$ *assigned to any two vertices* u *and* v *satisfy* $f(u) - f(v) = \chi$ *for some* $\chi = 1, 2$ *only if* $d(u, v) \geq 2 - \chi + 1$. *For* $d(u, v) > 2$, $f(u) \neq f(v)$.

Radio Labelling is equivalent to 2-coloring in the context of non-reusable frequencies. In particular, a radio labelling of an interference graph is an assignment of distinct labels/channels to all the transmitters, such that adjacent transmitters get labels at distance at least 2 (i.e. non-neighboring channels). The objective is to minimize the maximum label used (label span). Radio labelling is an appropriate model for practical applications, where the transmitters are not allowed to operate at the same channel (e.g. because they are in the same building or city), and a single value can be used for the spectral separation between the channels assigned to potentially interfering transmitters.

In this paper, we also focus on the radio coloring problem for planar graphs.

Definition 12. Radiocoloring *[30]: A radiocoloring of the planar graph* G *with* λ *colors is a function* $\Phi : V \to \mathbb{N}$ *such that* $|\Phi(u) - \Phi(v)| \geq 2$, *when* u, v *are neighbors in* G *and* $|\Phi(u) - \Phi(v)| \geq 1$ *when the minimum distance of* u, v *in* G *is 2.*

We shall concentrate on radiocoloring of planar graphs, since in most real life cases the actual network topologies are planar. Note also that many works in the relevant literature deal with various problems for the planar case [1, 18, 14, 23, 29, 49, 58, 61, 68, 65, 77, 78, 85, 95, 93].

For the reader's convenience we give below a table with the basic notation used in this paper.

TABLE OF NOTATION

Δ or $\Delta(G)$	: the maximum vertex degree of a graph G(V, E)
$\delta(G)$	: the minimum vertex degree of a graph G
$\Delta(v)$	: the degree of vertex v in G
$\Delta_k(v)$	: the degree of vertex v in G^k
$d(v, u)$	: the distance between vertex v and u in G
$\vartheta(G)$	: the thickness of a graph G
$ind(G)$	: the inductiveness of a graph G
$arb(G)$	: the arboricity of a graph G
$X_{order}(G)$	: the minimum number of integers needed in a radiocoloring of G
$X_{span}(G)$	: the minimum span needed in a radiocoloring of G
$X'_{order}(G)$	: the minimum number needed in a min span order radiocoloring of G
$X'_{span}(G)$	: the minimum span needed in a min order span radiocoloring of G
$\chi(G)$	: the vertex chromatic number of G
MIS	: Maximum Independent Set of G
MC_G	: the cardinality of the Maximum Clique in G
$TSP(1, 2)$	: the Traveling Salesman Problem with edge lengths one and two
$HP(1, 2)$	: the Hamiltonian Path problem with edge lengths one and two
$M(G, \lambda)$	: the Markov Chain with state space all radiocolorings of G with λ colors.

X_t, Y_t	: Markov Chains of type $M(G, \lambda)$
P^t	: the t-step transition probability matrix of a Markov Chain
$\delta_x(t)$	: variation distance (at time t, when starting from vertex x)
π	: the stationary distribution
$\tau_x(t)$	: the rate of convergence to stationarity with respect to vertex x

2 Previous Work

2.1 Hardness Results

In this section, we present some known results on the hardness of problems related (but different) to the ones we study in this paper. We do not provide full proofs of results of other authors in this section. We only give (hopefully) insightful comments and references.

The decision version of vertex coloring, which asks whether a given graph can be colored with k colors, is $\mathcal{NP}$-complete [35]. Recall that for some special families of graphs, such as the planar graphs, this question for $k \geq 4$ is proved to be true by the famous four color theorem. However, the question whether a planar graph can be colored with 3 colors remains $\mathcal{NP}$-complete.

The k-coloring problem is $\mathcal{NP}$-complete since it is a generalization of vertex coloring which is equivalent to k-coloring for k=1. Furthermore, the 2-coloring or radiocoloring problem for general graphs, and even for graphs of diameter two, is $\mathcal{NP}$-complete [29]. In that work, the problem was proved to be $\mathcal{NP}$-complete through a reduction from the Hamiltonian Path $(1,2)$ problem.

The $\mathcal{NP}$-completeness of distance-2-coloring

The problem of distance-2-coloring or coloring the square of graph G for general graphs was proved to be $\mathcal{NP}$-complete in the work of Lin and Skiena [67]. More accurately, the generalization of distance-2-coloring, distance-k-coloring, was proved to be $\mathcal{NP}$-complete. It can be easily seen this problem is equivalent to the problem of coloring the k-th power of a graph G, denoted G^k. The graph G^k has V as its vertex set and two vertices of G^k are adjacent if and only if there exists in G a path of length at most k between them. The work of Lin and Skiena proves that the problem of coloring G^k is also $\mathcal{NP}$-complete. This result implies the $\mathcal{NP}$-completeness of distance-k-coloring since the two problems are equivalent.

Theorem 1. *[67] Let a graph G. For any fixed integer $k \geq 1$, finding the minimum vertex coloring of its k-th power G^k is $\mathcal{NP}$-complete.*

The reduction of [67] works for distance-2-coloring as well. For this case, the reduction, from any graph $G(V, E)$ constructs a new graph $G'(V', E')$ defined as following. Any vertex of G is present in V' of G'; call these vertices *existing*. Additionally, each edge uv of G is replaced in G' with a new vertex i_{uv}, called *intermediate*. Then the vertex i_{uv} is connected with the vertices u and v, substituting the edge. Finally, all intermediate vertices are connected to each other. For an example, see Figure 1.

A careful study of this construction shows that G' is distance-2-colorable with $k + |E|$ colors if and only if G is colorable with k colors. To see this, observe that the insertion of intermediate vertices makes distance one constraints in G, distance two constraints in G'. Thus, the coloring of the vertices of V in G is equivalent to the distance-2-coloring of these vertices in G'. Furthermore, recall that the rest of the vertices of G', the so called intermediate vertices, are all connected to each other. Also, note that each of them is at distance at most two from any existing vertex. Therefore, we will need a set of new colors of size equal to their number for their distance-2-coloring. Since we add such a vertex for each edge of G, exactly $|E|$ new colors are needed for their distance-2-coloring. Concluding, if G is colored with k colors, then G' is colored with $k + |E|$ colors.

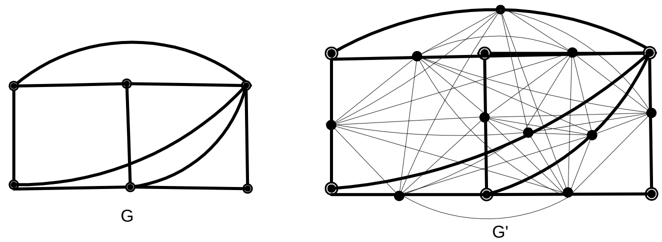


Fig. 1. The initial G and the new graph G' obtained from G according to the reduction of [67]. The new edges and vertices are shown in thinner lines and smaller points, respectively.

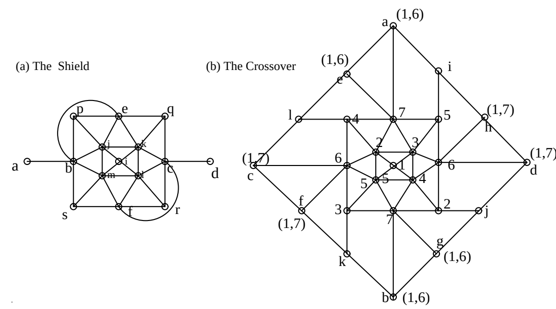


Fig. 2. The shield and the crossover used in the reduction of [84]

The problem of distance-2-coloring for planar graphs is studied in [84]. In that work, it is proved that the distance-2-coloring for planar graphs is $\mathcal{NP}$-complete.

Theorem 2. *[84] The distance-2-coloring problem is $\mathcal{NP}$-complete for planar graphs.*

The reduction of [84] is quite complicated. More specifically, it is proved that the *7-distance-2-coloring problem* is $\mathcal{NP}$-complete. This is the problem of deciding whether for a *planar* graph $G(V, E)$, there exist a mapping $c : V \rightarrow \{1, \dots 7\}$, such that for all pairs of distinct vertices u and v from V, if $c(u) = c(v)$, then the length of the shortest path between u and v is greater than 2. The problem is reduced from the 7-distance-2-coloring problem for general graphs which is known to be $\mathcal{NP}$-complete [71]. The reduction utilizes the well known technique of component design [35] and the "crossover" technique to eliminate crossings. Intuitively, the components, called crossover and shield (Figure 2), are designed to "carry" through a color from one side of a crossing to the other. Adjacent and distance-2 colors are glued "together" using a cascade of such components (Figure 3). This is achieved, very briefly, by constructing from a given graph G, a new graph G' as follows: Replace each node u of G by a "ring R^u" of $degree(u)$ shields and connect them in a ring and with a new central vertex c_u. Each crossing between nodes is replaced by shielded crossovers, connected in a proper way as demonstrated in Figure 4.

The $\mathcal{NP}$-completeness of the Hidden Terminal Interference Problem

In [8], a similar problem for *planar* graphs has been considered. This is the *Hidden Terminal Interference Avoidance (HTIA)* problem, which asks for coloring the vertices of a planar graph G with three colors so that vertices *at distance exactly 2* get different colors. In [8], this problem is shown to be $\mathcal{NP}$-complete.

The 3-Euclidean networks are planar networks where all vertices have at most 3-neighbors.

Theorem 3. *[8] The problem of deciding whether three colors are sufficient for Hidden Terminal Interference Avoidance in 3-Euclidean networks is $\mathcal{NP}$-complete.*

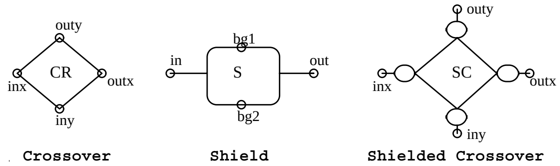


Fig. 3. The components and their pictorial abbreviation used in the reduction of [84]

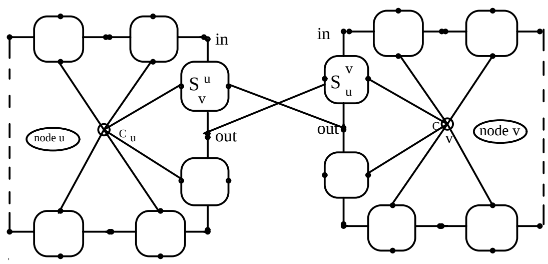


Fig. 4. Each ring represents a node, adjacent nodes are linked in the reduction of [67]

The construction reduces the 3-coloring of straight line planar graphs to HTIA for 3-Euclidean networks. A straight line planar graph is a graph whose edges can be represented by straight segments. Given a straight line graph $G(V, E)$, a 3-Euclidean network G' is constructed such that the vertices of G can be colored with three colors if and only if three different colors are sufficient to eliminate the hidden terminal interference in G'. The construction uses the component design technique.

We next present some previous work on polynomial time approximations to these problems.

2.2 Polynomial Time Approximations

λ-labeling

Bodlaender et all recently ([12]) studied a variation of FAP, called $\lambda(j, i)$-labeling. This is the problem of assigning colors to the vertices of G so that the colors assigned to any distance 1 vertices differ by at least j and the the colors assigned to any distance 2 vertices differ by at least i. The objective of such an assignment is to minimize the span. In [12], the authors presented lower bounds and approximations for $\lambda(2, 1)$-labeling, $\lambda(1, 1)$-labeling and $\lambda(0, 1)$-labeling for some interesting families of graphs: outerplanar graphs, graphs of treewidth k, bipartite, permutation and split graphs.

The problem of distance-2-coloring

The first work related to the problem of distance-2-coloring is that of Wegner [96], which focuses on the problem of coloring the square of a graph. He gave an instance for which the clique number is at least $3\Delta/2 + 1$ (which is the largest possible) and conjectured this to be an upper bound on the chromatic number for Δ large, where Δ is the maximum degree of the graph.

McCormick [71] gave a greedy algorithm that gives an $O(\sqrt{n})$-approximation for coloring the square of a general graph, where $n = |V|$. The algorithm starts with any ordering of the vertices of the network. The color assigned to vertex u is the smallest color that has not been used by any node within distance at most 2 from u.

The work of [84] provides a polynomial time approximation algorithm for the same problem for planar graphs with performance ratio of 9. The idea of the algorithm is quite simple: it is based on a "*largest-degree-first*" ordering of vertices. The analysis of the algorithm's performance is based on a property called *inductiveness* of graphs, which will be explained next.

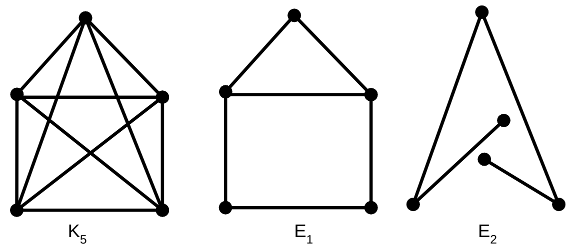


Fig. 5. The (nonplanar) graph K_5 and the partitioning of its edges into two planar sets of edges E_1, E_2.

Ramanathan and Loyd [84, 85] gave an approximation algorithm with performance guarantee of $O(\vartheta)$, where ϑ is the *thickness* of the graph. Intuitively, the thickness of the graph measures "its nearness to planarity", i.e. how planar the graph is. More formally, the thickness of a graph $G(V, E)$ is the minimum number of subsets into which the edge set E must be partitioned so that each subset in the partition forms a planar graph on V. For an example, remark that $\vartheta(K_5) = 2$, since its edge set can be partitioned into 2 subsets which are planar (and clearly K_5 itself is not planar). For a pictorial presentation see Figure 5. Thus, the performance of the algorithm obtained in [84, 85] may be expressed as a function of how planar the graph is.

Their algorithm is actually a modified version of another algorithm presented in the same work for the problem of *link scheduling*. Let (a, b) denote a directed edge from a to b. The *link scheduling problem* tries to color the edges of a graph G such that any pair of directed edges (a, b), (c, d) may be colored with the same color if and only if: (i) a, b, c and d are mutually distinct and $(a, d) \notin E$ and (ii) $(c, b) \notin E$ and $(c, d) \notin E$. The distance-2-coloring algorithm presented in [84, 85] is a modified version of the algorithm for the link scheduling problem, that colors the vertices of the graph (instead of the edges).

More specifically, the authors present a simple link scheduling algorithm for tree networks first. The tree topology of the network enables the partitioning of the graph into levels (of number equal to the tree's height) and the coloring of each level at a time. Next, this algorithm is generalized to networks of arbitrary topologies. This is achieved by a decomposition of the network into several pieces and applying a coloring, which is now easier, to each piece. The pieces are chosen to be *oriented graphs*. An *in-oriented* graph is one in which every vertex has at most one outgoing edge; thus the local view of a vertex shows several incoming edges but only one outgoing edge. An out-oriented graph is defined similarly. This property of oriented graphs enables a controllable coloring of each of them with restricted conflicts caused by the edges connecting such a subgraph with other subgraphs of the graph.

From the link scheduling algorithm for general networks, Ramanathan and Lloyd [84, 85] derived a distance-2-coloring algorithm by coloring the vertices of the graph (instead of the edges). The algorithm has performance ratio of $O(\vartheta)$ for general graphs. In planar graphs (where ϑ is equal to 1) the performance ratio is proved to be 9 and no better.

The distance-2-coloring for planar graphs algorithm presented in [85] is applicable to a more general class of graphs called *q-inductive* graphs and leads to a performance of at most $2q$ for such graphs. The *inductiveness* of a graph is defined as follows:

Definition 13. q-inductiveness *The inductiveness of a graph G, $ind(G)$, equals $max_{H \subseteq G}\{min_v(d_H(v))\}$, where H runs through all the subgraphs of G. Inductiveness leads to an ordering of the vertices $\{v_1, \cdots, v_n\}$ such that the pre-order of any v_i, $d^+(v_i) = |\{v_j \in N_G(v_i) : j > i\}|$, is at most q, i.e. the vertices of G can be assigned integers in such a way that each vertex is adjacent to at most q higher numbered vertices.*

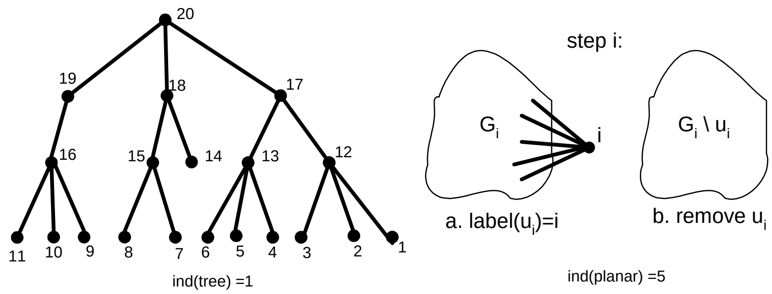


Fig. 6. A tree is 1-inductive and a planar graph is 5-inductive.

As an example, trees are 1-inductive. To see why consider a BFS ordering of the vertices of the tree. Then, starting from the last vertex and moving in reverse order, assign to vertices increasing numbers (see Figure 6). Obviously, each vertex is adjacent to at most one vertex with a higher number (its father). For another example, the outerplanar graphs are 2-inductive and planar graphs are 5-inductive. For an explanation of the 5-inductiveness of planar graphs, consider the following ordering of the vertices: Take a vertex u_1 of degree at most 5. Assign to this vertex number 1 and remove it from the graph. In the remaining graph ($G_1 = G \backslash u_1$), which is also planar, there is a vertex u_2 with degree at most 5. Assign to this vertex number 2 and then remove it from the graph, to get a new graph G_2. Continue in this way with the rest of the vertices of G_2. Since at each step we label a vertex of degree at most 5, there are at most 5 vertices adjacent to it with higher numbers (see Figure 6).

The work of Krumke, Marathe and Ravi ([62]) presents a 3-approximation algorithm for three classes of graphs: *(r,s)-civilized* graphs, planar graphs and graphs with bounded genus. The class of (r,s)-civilized graphs includes intersection graphs of circles whenever there is a (fixed) minimum separation between the centers of any pair of circles. More formally:

Definition 14. (r,s)-civilized graphs: *A graph belongs to the class of (r,s)-civilized graphs if for each real $r > 0$ and $s > 0$, there exists a positive integer $d \geq 2$ such that the graph can be drawn in $\mathbb{R}^{\mathrm{d}}$ so that the length of each edge is at most r and the distance between any two points is at least s.*

This class of graphs is a reasonable model for several classes of packet radio networks. To see this, consider packet radio networks in which the range of any transmitter can be considered as a circular region with the transmitter at the center of the circle. Let r the radius of the region corresponding to a transmitter's maximum range. Further, it is natural to assume a minimum separation s between any pair of transmitters, since the equipment carrying the transmitters cannot be colocated.

The general idea behind the coloring algorithms presented in the work of [62] is the following: Given a graph, suitably partition its vertices into *levels*, similar to the partitioning of [85] explained briefly above. For planar and related classes of graphs, this is done by choosing an arbitrary vertex v and constructing a breadth-first search (BFS) spanning tree T rooted at v. Each vertex u can now be assigned a unique level number, which is the number of nodes in the unique path in T from the root to u, including the end points.

This method of assigning levels to nodes has the property that for any level i, no node at level i is adjacent to a node at level $i + 2$ or greater. This property is exploited in the algorithms presented through reusing the colors of vertices used at layer i at levels $i + 3$ or greater. A lemma from [62], stating that for a given G of treewidth k and maximum degree Δ, the treewidth of G^2 is at most $(k + 1)\Delta - 1$, is used to prove that the distance-2-coloring problem can be solved polynomially for graphs with constant treewidth and maximum degree.

The same algorithmic approach is used in (r, s)-civilized graphs to give a 3-approximation algorithm for distance-2-coloring. More specifically, the graph is divided into horizontal levels (strips) as before.

The strips are divided into three groups so that the levels of the same group are located apart from each other enough to allow taking the same set of colors. Furthermore, let S_i denote the vertices of G in strip i. It is shown that each induced subgraph of G^2 induced on each subset S_i is of bounded treewidth and degree; thus, it can be optimally colored in polynomial time. These observations are utilized to prove that the above coloring method leads to a 3-approximation for the problem of distance-2-coloring for planar (r,s)-civilized graphs with constant r and s.

The same algorithmic approach applied in planar graphs of bounded degree leads to a 3-approximation algorithm. (Agnarsson and Halldórsson [1] showed recently that a 2-approximation follows easily from Krumke, Marathe and Ravi's approach [62].)

Further Exploitation of Graph Inductiveness

The recent work of Agnarsson and Halldórsson [1] studies the problem of coloring powers of graphs. Recall that the k-th power of a graph G, G^k, is the graph defined on the same set of vertices that has an edge between any two vertices if their distance in G is at most k. This work exploits also the property of inductiveness of a graph. A q-inductive graph can be colored with q colors by ordering the vertices according to a q inductive ordering and coloring them sequentially based on this order in a greedy manner. Since the graph is q-inductive, it can be easily shown that this greedy algorithm approximates distance-2-coloring by a ratio of $2q$.

In [1], the authors show that the inductiveness and the chromatic number of a planar graph G^k is $O(\Delta^{k/2})$, which is tight. These bounds lead to a 2-approximation algorithm for coloring the squares of planar graphs.

They first bound inductiveness of a power graph G^k. In order to show that the power of a graph is q-inductive, where q is necessarily a function of the maximum degree Δ, they show the existence of a vertex $v \in V(G^k) = V(G)$ such that

- $\Delta_k(v) \leq q$, where $\Delta_k(v)$ is the degree of vertex v in G^k and
- v has a neighbor u such that $\Delta(u) + \Delta(v) - 2 \leq \Delta$, where $\Delta(v)$ is the degree of vertex v in G.

In order to bound the inductiveness of a planar graph the following lemma is utilized:

Lemma 1. *(Agnarsson and Halldórsson, [1]) Let G be a simple planar graph of maximum degree $\Delta \geq 26$. Then there exists a vertex $v \in V(G)$ satisfying one of the following:*

1. *$\Delta(v) \leq 25$ and at most one neighbor of v has degree ≥ 26.*
2. *$\Delta_2(v) \leq \lfloor \frac{9\Delta}{5} \rfloor + 1$ and only two neighbors of v have degree ≥ 26, where $\Delta_i(v)$ is the degree of vertex v in G^i.*

This lemma is used to prove:

Theorem 4. *(Agnarsson and Halldórsson, [1]) If G is planar with maximum degree $\Delta \geq 749$, then G^2 is $\lfloor \frac{9\Delta}{5} \rfloor + 1$-inductive.*

Proof. Assume that $\Delta \geq 25+25-2$ and that we have a vertex v of G which satisfies the first condition of Lemma 1 (i.e $\Delta(v) \leq 25$ and v has at most one neighbor of degree ≥ 26). If v has a neighbor u of degree $\Delta(u) \leq 25$, then

$$\Delta_2(v) \leq 24(25) + \Delta = 600 + \Delta$$

and

$$\Delta(v) + \Delta(u) - 2 \leq 25 + 25 - 2 \leq \Delta$$

If v has no neighbor of degree at most 25, then it has only one neighbor u. In this case, $\Delta_2(v) \leq \Delta$ since v has only one neighbour u ($\Delta(v) = 1$), and $\Delta(u) \geq 26$. Moreover, $\Delta(v) + \Delta(u) - 2 \leq 1 + \Delta - 2 \leq \Delta$.

In the proof of Lemma 1 (see [1]), it was assumed that there is no vertex in V_l with at most one neighbor of V_h. In that case, there is a vertex of G, called z_3, with $\Delta_2(z_3) \leq \lfloor \frac{9\Delta}{5} \rfloor + 1$. Also, z_3 has at most two neighbours in V_l. If z_3 has no neighbors in V_l, then, since the only neighbors of z_3 in V_h are u and v, we have $\Delta(z_3) + \Delta(v) - 2 = \Delta(z_3) + \Delta(u) - 2 < \Delta$. If z_3 has a neighbor in V_l, say z_1, then $\Delta(z_3) + \Delta(z_1) - 2 \leq \Delta$.

In any case we see that we can always find a vertex w of G with $\Delta_2(w) \leq max \left\{600 + \Delta, \lfloor \frac{9\Delta}{5} \rfloor + 1\right\}$, and such that w has a neighbor w' with $\Delta(w) + \Delta(w') - 2 \leq \Delta$. By Lemma 1 we conclude that G^2 is $\lfloor \frac{9\Delta}{5} \rfloor + 1$-inductive.

□

The bound is proved to be tight by giving an instance G whose square, G^2, is of minimum degree $\lfloor \frac{9\Delta}{5} \rfloor + 1$.

Furthermore, the work of [1] studied the chromatic number of general powers of planar graphs. The following theorem was proved:

Theorem 5. *(Agnarsson and Halldórsson, [1]) Let G be a planar graph with maximum degree Δ. For any integer $k \geq 1$, G^k is $O(\Delta^{\lfloor k/2 \rfloor})$-colorable. Also, there is a family of graphs that attains this bound. This bound is also asymptotically tight for the clique number, inductiveness, arboricity (defined below), and the minimum degree of G^k.*

The proof of the theorem involves among others the *arboricity* of a graph. The *arboricity* of a graph G is

$$arb(G) = max_{H \subseteq G} \left\lceil \frac{|E(H)|}{V(H) - 1} \right\rceil$$

By the Nash-Williams Theorem ([76]), there are $arb(G)$ edge-disjoint subforests of G that cover all the edges of G. Arboricity is closely related to inductiveness. It is proved that:

Lemma 2. *(Agnarsson and Halldórsson, [1]) For any graph G, we have $arb(G) \leq ind(G) \leq 2\, arb(G)$.*

Thus, in order to bound the inductiveness of G^k we may bound the arboricity of G^k. Thus, we want to show that there is a sequence $(\alpha_k)_{k=1}^{\infty}$ such that

$$arb(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}.$$

Using some technical ideas and the Nash-Williams Theorem it is proved that:

Lemma 3. *[1] If G be a planar graph with maximum degree Δ and $k \geq 1$ an integer, then we have $arb(G^k) \leq \alpha_k \Delta^{\lfloor k/2 \rfloor}$, where $\alpha_1 = 3$, $\alpha_2 = 9$ and $\alpha_k = 4^{k-1} 10^{k^2/4}$*

By Lemma 2 we get the following corollary:

Corollary 1. *(Agnarsson and Halldórsson, [1]) For a simple planar graph G of maximum degree Δ and an integer $k \geq 1$, we have that G^k is $2\alpha_k \Delta^{\lfloor k/2 \rfloor}$-inductive.*

The previous lemmas are utilized to complete the proof of theorem 5.

3 Overview of this paper

In this work, we focus on two important frequency assignment problems: radio labelling and radiocoloring (for their definitions see section 1).

For the radio labelling problem:

– We show that radio labelling remains $\mathcal{NP}$-complete even for graphs of diameter 2. Furthermore, radio labelling is MAX–SNP-hard and approximable in polynomial time within $\frac{7}{6}$.

– We present a polynomial time algorithm that computes an optimal radio labelling of a graph, given a coloring with a constant number of colors. Thus, we prove that radio labelling is in $\mathcal{P}$ for planar graphs, and for graphs colorable with a constant number of colors, in polynomial time.
– We present an efficient $\mathcal{NC}$ algorithm that approximates radio labelling within a factor of $\frac{3}{2}$. This algorithm outperforms all known sequential approximation algorithms, when restricted to graphs with chromatic number (clique number) less than $\frac{|V|}{6}$.
– We prove that there exists a constant $\Delta^\star < 1$, such that radio labelling for graphs $G(V,E)$ with maximum degree $\Delta(G) \in [\Delta^\star|V|, |V|)$ is essentially as hard to approximate as for general graphs. On the other hand, if $\Delta(G) < \frac{n}{2}$, then an optimal radio labelling of value $|V|$ can be efficiently computed.

For the radiocoloring problem for planar graphs:

– We prove that various important versions of radiocoloring remain $\mathcal{NP}$-complete for planar graphs.
– We provide an $O(n \max\{\log n, \Delta\})$ time algorithm ($|V| = n$) which obtains a radiocoloring of a planar graph $G(V,E)$ that *approximates the minimum number of colors needed within a ratio which tends to 2* as the maximum vertex degree Δ of G increases.
– We provide a fully polynomial randomized approximation scheme (FPRAS) for the number of radiocolorings of a planar graph G with λ colors, for the case where $\lambda \geq 4\Delta + 50$.

This work is organized in the following way. In section 4 we deal with the Radio Labelling problem. More specifically, after a summary of results in section 4.1, in section 4.2 we prove that the Radio Labelling problem is $\mathcal{NP}$-complete for general graphs. In section 4.3 we provide an optimal polynomial time algorithm for the Radio Labelling of a planar graph, thus showing that the problem is in $\mathcal{P}$ for planar graphs. In section 4.4 we give an efficient $\mathcal{NC}$ approximation algorithm for Radio Labelling of arbitrary graphs. In section 4.5 we discuss the Radio Labelling problem in graphs of bounded maximum degree.

In section 5 we study the Radiocoloring problem. First, we give in section 5.1 a summary of related previous work and our results. In section 5.2 we discuss the difference (despite the similarities) of the radiocoloring problem and the problem of coloring the square of a graph. We then prove in section 5.3 the $\mathcal{NP}$-completeness of the Radiocoloring problem, while in section 5.4 we provide an efficient approximation algorithm for the problem. In section 5.5 we give a fully polynomial time randomized approximation scheme for the number of radiocolorings, with a given number of colors, of a planar graph. Finally, we discuss in section 6 some open problems and directions for further research.

Preliminary versions of these results have appeared in [27–32].

4 Radio Labeling

Radio labeling is the variant of radio coloring that does not allow any frequency reuse. In particular, radio labeling is the problem of assigning distinct integer channels (labels) to transmitters such that adjacent transmitters get channels differing by at least 2 (i.e., non-neighboring channels). The objective is to minimize the maximum label used (label span). The definition of radio labeling has been communicated to us by [45]. Radio labeling is an appropriate model for practical applications, where the transmitters are not allowed to operate at the same channel, and a single channel can be used for the spectral separation between channels assigned to potentially interfering transmitters. Radio labeling has been studied in [27]. Competitive algorithms and lower bounds for on-line radio labeling are described in [29].

Radio Labeling and related coloring problems that model the assignment of non-reusable frequencies are widely used for obtaining lower bounds on the optimal values of general FAP instances. Since FAP is an intractable optimization problem and approximation algorithms of guaranteed quality are not known, lower bounds are necessary in order to assess the quality of the assignments found by heuristic algorithms. Two approaches are mainly used for obtaining lower bounds [89]. The first approach is based on a transformation from frequency assignment to the Travelling Salesman Problem (TSP) [86]. In general, lower bounds obtained from TSP are poor, because the corresponding instances contain many edges of zero length, and many interference constraints are violated. The second approach is based on the observation that certain subgraphs (such as kernels [6]) of the interference graph determine the value of the frequency span. Moreover, the majority of vertices of such subgraphs are assigned distinct channels. Thus, good lower bounds can be obtained by computing near optimal radio labelings for such subgraphs.

4.1 Summary of Results for Radio Labeling

We show that radio labeling and radio coloring remain $\mathcal{NP}$-complete for graphs of diameter 2. We also prove that radio labeling is equivalent to the problem of finding a spanning path of minimum length in a complete graph, where the edge lengths are either 1 or 2 (HP(1,2)). Obviously, HP(1,2) is the path analogue of TSP(1,2). Therefore, radio labeling is MAX–SNP-hard and approximable in polynomial time within $\frac{7}{6}$ [79]. In a recent survey [36], the relation between radio labeling and path covering is also observed.

Then, we present a polynomial time algorithm that computes an optimal radio labeling of a graph, given a coloring with a constant number of colors. Thus, we prove that radio labeling is in $\mathcal{P}$ for planar graphs, and for graphs colorable with a constant number of colors in polynomial time. Although planar interference graphs are quite typical in the context of FAP, this is the very first result concerning the complexity of radio labeling for planar graphs.

We present an efficient $\mathcal{NC}$ approximation algorithm which, given an arbitrary graph $G(V, E)$, computes a radio labeling of value at most $|V|$ plus the cardinality of the Maximum Clique of G. Hence, this algorithm approximates radio labeling within a factor of $\frac{3}{2}$. Given a graph $G(V, E)$ with chromatic number $\chi(G)$, our $\mathcal{NC}$ algorithm operates without assuming a near optimal coloring or any knowledge on $\chi(G)$, and it computes a radio labeling of value within an additive term of $\chi(G)$ from the optimal. Therefore, it outperforms all known sequential approximation algorithms, when restricted to graphs with chromatic number (clique number) less than $\frac{|V|}{6}$. This is important because typical interference graphs have small chromatic number. On the other hand, if we exclude some classes of graphs (e.g. planar), we do not know how to efficiently compute an approximate coloring with a reasonable number of colors, even for graphs with constant chromatic number (cf. [53]). Consequently, our $\mathcal{NC}$ approximation algorithm complements the aforementioned exact algorithm, which assumes the existence of a near optimal coloring.

Additionally, we prove that there exists a constant $\Delta^\star < 1$, such that radio labeling for graphs $G(V, E)$ with maximum degree $\Delta(G) \in [\Delta^\star|V|, |V|)$ is essentially as hard to approximate as for general graphs. On the other hand, if $\Delta(G) \in (0, n/2)$, then an optimal radio labeling can be efficiently computed.

Since radio labeling is closely related to HP(1,2) and TSP(1,2), the results above can be translated to similar results for the latter problems. In particular, given a partition of a graph with a constant number of cliques, we show how to decide in polynomial time if this graph is Hamiltonian. We are not aware of a similar algorithm that exploits partition into cliques so as to decide Hamiltonicity. Also, our $\mathcal{NC}$ algorithm approximates TSP(1,2) and HP(1,2) within a factor of $\frac{3}{2}$. This is the first $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for TSP(1,2). Up to now, the best known parallel algorithm for metric TSP and all the special cases is the algorithm of Christofides (an $\mathcal{RNC}$ implementation for metric TSP is described in [21]). However, since this algorithm requires the computation of a perfect matching of minimum weight (maximum cardinality matching for the case of TSP(1,2)), it is in $\mathcal{RNC}$, but it is not known to be in $\mathcal{NC}$. Additionally, we show that dense instances of TSP(1,2) and HP(1,2) are essentially as hard to approximate as general instances; this is in sharp contrast to the success in approximating dense instances of many combinatorial optimization problems (see [54] for a survey). Very recently, a different proof of the same result appears independently in [33].

4.2 Preliminaries

Given a graph $G(V,E)$, $\chi(G)$ denotes the chromatic number of G, MIS_G is the cardinality of the Maximum Independent Set, MC_G is the cardinality of the Maximum Clique, and $d(u,v)$ denotes the length of the shortest path between $u, v \in V$. Clearly, for all graphs G, $\mathrm{MC}_G \leq \chi(G)$. The complementary graph $\overline{G}$ is a graph on the same vertex set V that contains an edge $uv \in V \times V$, iff $uv \notin E$. Clearly, $\mathrm{MIS}_G = \mathrm{MC}_{\overline{G}}$. A graph $G(V,E)$ is called *δ-dense* for some constant δ such that $1 > \delta > 0$, if the minimum degree $\delta(G)$ is at least $\delta|V|$. A graph $G(V,E)$ is called *$(1-\delta)$-bounded* for some constant δ such that $1 > \delta > 0$, if it is the complement of a δ-dense graph. A $(1-\delta)$-bounded graph $G(V,E)$ has maximum degree $\Delta(G)$ bounded from above by $(1-\delta)|V|$, i.e., no vertex of G is connected to more than a $(1-\delta)$ fraction of the others.

In the *Traveling Salesman Problem* (TSP), we are given n nodes, and for each pair of distinct nodes i and j a distance $d_{i,j}$. The objective is to find a salesman tour, i.e., a simple cycle that visits each node exactly once of minimum length. TSP with distances one and two (TSP(1,2)) is a special case of TSP restricted to complete graphs, where all edge lengths are either 1 or 2; clearly, TSP(1,2) is a special case of metric TSP, since the edge lengths always satisfy the triangle inequality. Alternatively, TSP(1,2) is a generalization of the Hamiltonian Cycle problem, where each edge of the input graph has length 1, and each non-edge has length 2. In this case, we seek the tour (simple spanning cycle) with the fewest possible non-edges. An interesting variant of TSP(1,2) is the Hamiltonian Path with distances 1 and 2 (HP(1,2)), where we seek a simple spanning path of minimum possible length. Obviously, both TSP(1,2) and HP(1,2) are $\mathcal{NP}$-complete, as generalizations of the Hamiltonian Cycle and Path problems. Moreover, they are MAX–SNP-hard, even if the graph formed by the edges of length 1 has maximum degree at most 4, and there exists a polynomial time $\frac{7}{6}$-approximation algorithm [79]. In the sequel, the graphs formed by the edges of length 1 are sometimes used for defining instances of TSP(1,2) and HP(1,2).

Definition 15. **κ-labelling***: Given a graph $G(V,E)$ and an integer $k \geq 2$, the problem of* κ-labelling *is to compute a function $\lambda_\kappa : V \mapsto \{1, \ldots, \nu\}$, such that, for all $v, u \in V$, $|\lambda_\kappa(v) - \lambda_\kappa(u)| \geq \left\lceil \frac{\kappa}{d(v,u)} \right\rceil$. The* value *of a κ-labelling is the maximum label used (label span).*

This section is devoted to the study of 2-labelling, which is also called *radio labelling* or *distance-2 labelling*. We start by proving that radio labelling is equivalent to HP(1,2) in the complementary graph.

Radio labelling can be thought of as a vertex arrangement problem. In particular, given a vertex ordering $v_1, v_2, \ldots, v_n$, a radio labelling can be computed as follows: $\lambda_2(v_1) = 1$, and for $i = 1, \ldots, n-1$, $\lambda_2(v_{i+1}) = \lambda_2(v_i) + 1$, if $\{v_i, v_{i+1}\} \notin E$, and $\lambda_2(v_{i+1}) = \lambda_2(v_i) + 2$, otherwise. Conversely, any radio labelling implies a vertex arrangement, in the sense that v precedes u, iff $\lambda_2(v) < \lambda_2(u)$. Throughout this paper, we only consider radio labellings that are *minimal* with respect to the corresponding vertex arrangement; thus, given a radio labelling L, there does not exist another radio labelling L' that corresponds to the same vertex arrangement with $\lambda_2(L') < \lambda_2(L)$. In fact, the following is an immediate consequence of the discussion above.

Lemma 4. *Radio labelling is equivalent to HP(1,2) in the complementary graph.*

Proof. Given an instance of radio labelling, (that is, a graph $G(V,E)$), the corresponding instance of HP(1,2) is a complete graph $\hat{G}$ on the vertex set V, and the distance function $\hat{d}$ is defined for all

$v, u \in V$, $v \neq u$, by

$$\hat{d}(v,u) = \begin{cases} 1 & \text{if } (v,u) \notin E \\ 2 & \text{if } (v,u) \in E \end{cases}$$

Additionally, given a radio labelling of value $\nu \geq n$, we can easily compute a spanning path of $\hat{G}$ of length exactly $\nu - 1$ by considering the vertices in increasing order of their labels. We claim that the length of this spanning path up to any vertex of label i is exactly $i - 1$. We prove the claim by induction on i.

The claim is clearly true for the first vertex ($i = 1$). We assume that it is true for any vertex v of label i, and let u be the next vertex in the spanning path. If the label of u is $i + 1$, the edge vu is not present in G, and $\hat{d}(v,u) = 1$. Thus, the length of the path up to the vertex u is exactly i. If the label of u is $i + 2$, there does not exist a vertex of label $i + 1$. Therefore, $vu \in E$, and $\hat{d}(v,u) = 2$. Consequently, the path up to u has length exactly $i + 1$. Obviously, the resulting spanning path has length exactly $\nu - 1$.

Conversely, given an instance $(\hat{G}, \hat{d})$ of HP(1,2), we construct an instance $G(V,E)$ of radio labelling by only connecting the vertex pairs that are at distance 2. Furthermore, given a spanning path of $\hat{G}$ of length $\nu - 1 \geq n - 1$, we can easily compute a radio labelling of G of value exactly ν, as follows: We select (arbitrarily) an end vertex v of the path, and we assign the label 1 to v. Then, the label of each vertex $u \in V$ is one plus the distance of u from v in the spanning path. Alternatively, if the label $\lambda_2(w)$ is assigned to a vertex w, and u is the next vertex in the spanning path, the label of u is $\lambda_2(u) = \lambda_2(w) + \hat{d}(w,u)$. Clearly, the last vertex of the path gets the label ν, and this procedure assigns unique labels from the set $\{1, \ldots, \nu\}$ to the vertices of G. Since an edge $\{v,u\}$ is present in E, iff $\hat{d}(v,u) = 2$, the vertices that are adjacent in E are at a distance at least 2 in the spanning path. Therefore, if $\{v,u\} \in E$, then $|\lambda(v) - \lambda(u)| \geq 2$, and the resulting labeling is a radio labeling. □

Lemma 4 implies that radio labelling is $\mathcal{NP}$-complete, and MAX–SNP-hard, even if the minimum degree of the input graph $G(V,E)$ is at least $|V| - 5$. Moreover, it is known to be approximable in polynomial time within $\frac{7}{6}$ [79]. On the other hand, given a graph $G(V,E)$ with maximum degree $\Delta(G) < \frac{|V|}{2}$, a radio labelling of value $|V|$ can always be computed in polynomial time, because of Dirac's Theorem. (e.g., see [46], Corollary 7.3(b)).

Radio labelling is the analogue of radio coloring in the context of non-reusable frequencies. Given a coloring of a graph $G(V,E)$ with χ colors, it is easy to find a radio labelling of value at most $|V| + \chi - 1$. Therefore, $\lambda_2(G) \leq |V| + \chi(G) - 1$. However, it usually is $\mathcal{NP}$-hard even to approximate the value of $\chi(G)$ within a reasonable factor. Additionally, for graphs G of diameter 2, radio labelling is equivalent to radio coloring, and both problems are $\mathcal{NP}$-complete for such graphs, even though coloring G^2 is trivial.

Lemma 5. *Radio labelling and radio coloring restricted to graphs with diameter 2 are $\mathcal{NP}$-complete.*

Proof. Clearly, a radio coloring of a graph $G(V,E)$ of diameter 2 must assign distinct labels to all the vertices of G. Therefore, the problem of radio coloring a graph G of diameter 2 is equivalent to the problem of radio labeling G.

Since radio labelling is in $\mathcal{NP}$, and it is equivalent to Hamiltonian Path in the complementary graph (see also the proof of Lemma 4), we conclude the proof by showing that Hamiltonian Path remains $\mathcal{NP}$-complete for complements of graphs of diameter 2.

Let $G'(V', E')$ be any graph, and consider any two non-adjacent vertices $s, t \in V'$. The problem of deciding if G' contains a Hamiltonian Path, that starts from s and ends to t is known to be $\mathcal{NP}$-complete (Hamiltonian Path between Two Vertices [35]). Let $G^{(c)}(V' \cup \{v_s, v_t\}, E' \cup \{(s, v_s), (t, v_t)\})$ be the graph obtained from G' by adding two independent vertices v_s, v_t, and connecting v_s to s and v_t to t (see also Figure 7). Deciding if $G^{(c)}$ contains a Hamiltonian Path is $\mathcal{NP}$-complete, because $G^{(c)}$ contains a Hamiltonian Path iff G' contains a Hamiltonian Path between s and t. Moreover, the following observations for the complementary graph $\overline{G^{(c)}}$ justify that $\mathrm{diam}(\overline{G^{(c)}}) = 2$.

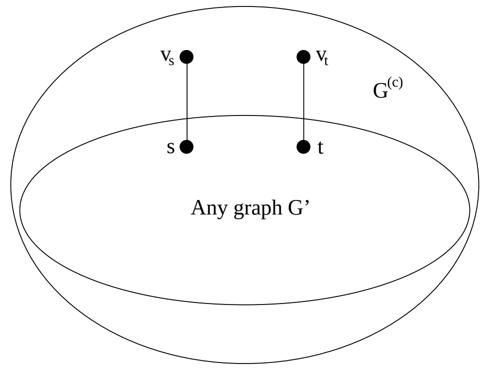


Fig. 7. The graph $G^{(c)}$ whose complement has diameter 2.

1. The vertex pairs $(s,t),(v_s,v_t),(s,v_t)$ and (t,v_s) are connected by edges.
2. Any pair of vertices $u,w \in V' - \{s,t\}$ are at distance at most two, because they are connected to both v_s and v_t.
3. Any vertex $u \in V' \cup \{v_s\} - \{s,t\}$ is at distance at most two from s, because both u and s are connected to v_t.
4. Any vertex $u \in V' \cup \{v_t\} - \{s,t\}$ is at distance at most two from t, because both u and t are connected to v_s. □

4.3 An Exact Algorithm for Constant Number of Color Classes

Next, we show that, given a graph $G(V,E)$, with $|V| = n$, and a coloring of G with a constant number of colors, we can decide in polynomial time if there exists a radio labelling for G of value n. We also show how to use this decision procedure in order to compute an optimal radio labelling of G in polynomial time.

The class consisting of the graphs colorable with a constant number of colors in polynomial time is non-trivial. An interesting subclass of it is the class of planar graphs. The class of planar graphs is very important for the FAP, because the interference graphs are planar in many practical applications. Moreover, the complexity of finding an optimal radio labelling in such graphs is open, despite the fact that a near optimal radio labelling can be easily obtained from a coloring with a constant number of colors.

Radio labelling is equivalent to Hamiltonian Path in the complementary graph, and a coloring with a constant number of colors corresponds to a partition of the complementary graph into a constant number of cliques. In the following, the technical part of the proof is presented by means of Hamiltonian paths (actually cycles) in the complementary graph. The main reason is that well understood graph theoretic tools (e.g., spanning trees, Eulerian trails) can naturally be applied to the context of Hamiltonian cycles and partitions into cliques. As a result, the arguments are better motivated and the presentation is more consistent. Another (technical) reason is that we are able to show how to decide in polynomial time if the complementary graph contains a Hamiltonian cycle. Notice that, given a graph and a Hamiltonian path, it is $\mathcal{NP}$-complete to decide if the graph is Hamiltonian (cf. [35]).

Hamiltonian cycles and Partitions into Cliques Throughout this section, let $G(V,E)$ be a connected graph, $\kappa > 1$, and $\mathcal{C} = \{C_1, C_2, \ldots, C_\kappa\}$ be a partition of V into κ cliques.

A set of inter-clique edges (i.e., edges connecting vertices belonging to different cliques of $\mathcal{C}$) $M \subseteq E$ is an HC-set if M can be extended to a Hamiltonian cycle using only clique-edges, i.e. there exists a $M^{(c)} \subseteq E$ of clique edges such that $M \cup M^{(c)}$ is a Hamiltonian cycle.

We first show that, given a set of inter-clique edges M, we can decide if M is an HC-set and construct a Hamiltonian cycle from M in poly(n, κ) time (Proposition 1). Then, we prove that G is Hamiltonian iff there exists an HC-set of cardinality at most $\kappa(\kappa-1)$ (Lemmas 6 and 7). The algorithm exhaustively searches all the sets of inter-clique edges of cardinality at most $\kappa(\kappa-1)$ for an HC-set. Additionally, we conjecture that, if G is Hamiltonian, then there exists an HC-set of cardinality at most $2(\kappa-1)$. We prove this conjecture for a special case (Lemma 8) and we use Lemma 6 to show the equivalence to Conjecture 1.

Let $\mathcal{C} = \{C_1, \ldots, C_\kappa\}$ be a partition of V into $\kappa > 1$ cliques. Given a set $M \subseteq E$ of inter-clique edges, the *clique graph* $T(C, M)$ contains exactly κ vertices, that correspond to the cliques of $\mathcal{C}$, and represents how the edges of M connect the different cliques. If M is an HC-set, then the corresponding clique graph $T(C, M)$ is connected and Eulerian. However, the converse is not always true.

Given a set of inter-clique edges M, we color an edge RED, if it shares a vertex of G with another edge of M. Otherwise, we color it BLUE. The corresponding edges of G are colored with the same colors, while the remaining edges $(E \backslash M)$ are colored BLACK. Additionally, we color RED each vertex $v \in V$, which is the common end vertex of two or more RED edges. We color BLUE each vertex $v \in V$ to which exactly one edge of M (RED or BLUE) is incident. The remaining vertices of G are colored BLACK (Figure 8).

Let H be any Hamiltonian cycle of G and let M be the corresponding set of inter-clique edges. Clearly, RED vertices cannot be exploited for visiting any BLACK vertices belonging to the same clique. If H visits a clique C_i through a vertex v, and leaves C_i through a vertex u, then $v, u \in C_i$ consist a BLUE vertex pair. A BLUE pass through a clique C_i is a simple path of length at least one, that entirely consists of non-RED vertices of C_i. A clique C_i is covered by M, if all the vertices of C_i have degree at most two in M, and the existence of a non-RED vertex implies the existence of at least one BLUE vertex pair. The following proposition characterizes HC-sets.

Proposition 1. *A set of inter-clique edges M is an HC-set iff the corresponding clique graph $T(C, M)$ is connected, Eulerian, and,*

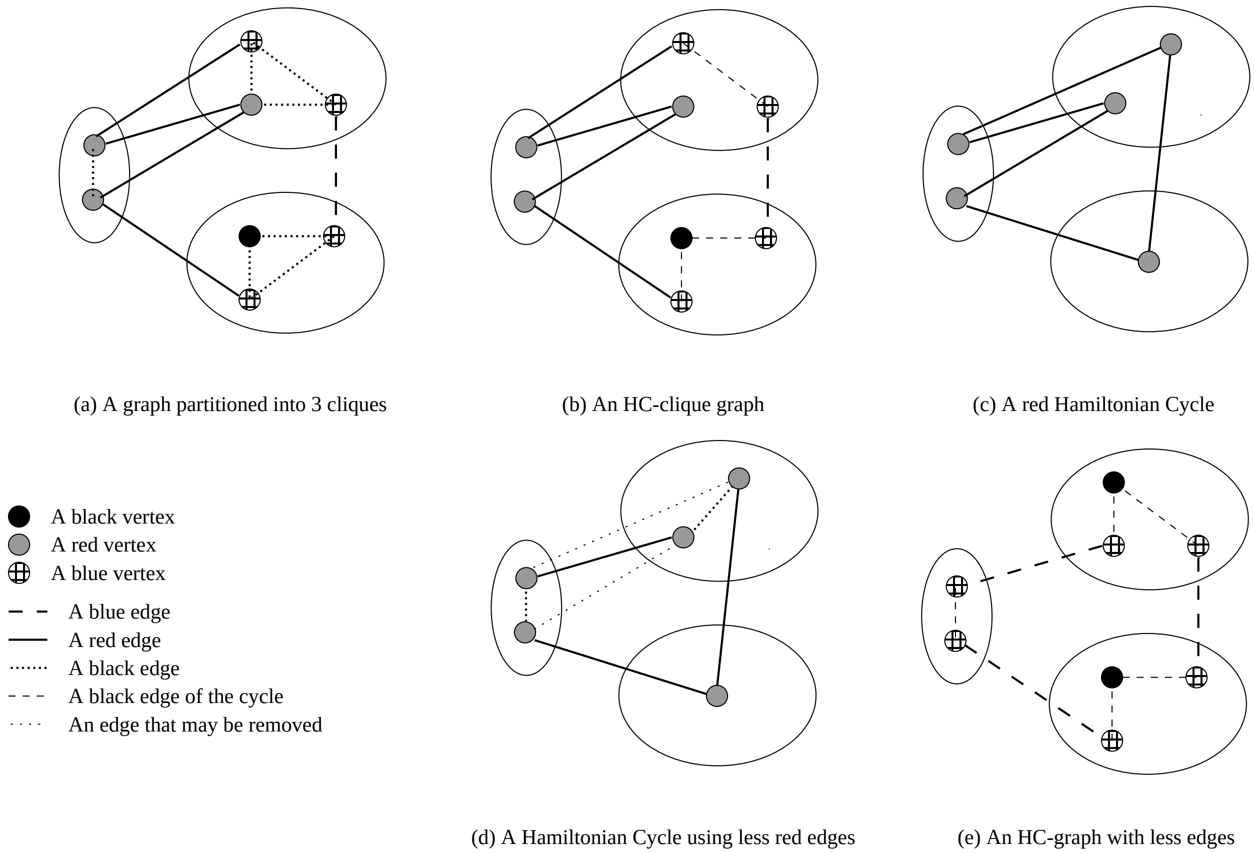
(a) For all $i = 1, \ldots, \kappa$, C_i is covered by M; and
(b) There exists an Eulerian trail R for T such that: For any RED *vertex $v \in V$, R passes through v exactly once using the corresponding* RED *edge pair.*

Proof. Clearly, any HC-set corresponds to a connected, Eulerian clique graph $T(C, M)$ that fulfils both (a) and (b). Conversely, we can extend M into a Hamiltonian cycle H following the Eulerian trail R. H passes through any RED vertex exactly once, because of (a) and (b). Moreover, since R is an Eulerian trail and (a) holds for M, H passes through all the BLUE and the BLACK vertices exactly once Therefore, H is a Hamiltonian cycle. □

The proof of Proposition 1 implies a deterministic procedure for deciding if a set of inter-clique edges is an HC-set in poly(n, κ) time. Moreover, in case that M is an HC-set, this procedure outputs a Hamiltonian cycle.

Lemma 6. *Let $B_\kappa \geq 2$ be some integer only depending on κ such that, for any graph $G(V, E)$ and any partition of V into κ cliques, if G is Hamiltonian and $|V| > B_\kappa$, then G contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges* (RED *vertices*).
Then, for any graph $G(V, E)$ and any partition of V into κ cliques, G is Hamiltonian iff it contains a Hamiltonian cycle with at most B_κ inter-clique edges.

Proof. Let H be the Hamiltonian cycle of G containing the minimum number of inter-clique edges and let M be the corresponding set of inter-clique edges. Assume that $|M| > B_\kappa$. The hypothesis implies that H cannot entirely consist of RED vertices. Therefore, H should contain at least one BLUE vertex pair.



(a) A graph partitioned into 3 cliques (b) An HC-clique graph (c) A red Hamiltonian Cycle

(d) A Hamiltonian Cycle using less red edges (e) An HC-graph with less edges

Fig. 8. An application of Lemmas 6 and 7.

We substitute any BLUE pass of H through a clique C_i with a single RED super-vertex $\hat{v}$, that also belongs to the clique C_i. Hence, $\hat{v}$ is connected to all the remaining vertices of C_i using BLACK edges. These substitutions result in a cycle H' that entirely consists of RED vertices, and contains exactly the same set M of inter-clique edges with H.

Clearly, the substitutions of all the BLUE passes of H with RED super-vertices result in a graph $G'(V', E')$ that is also Hamiltonian, $|V'| > B_\kappa$, and V' is partitioned into κ cliques. Moreover, for any Hamiltonian cycle $H_{G'}$ of G', the reverse substitutions of all the RED super-vertices $\hat{v}$ with the corresponding BLUE passes result in a Hamiltonian cycle of G that contains exactly the same set of inter-clique edges with $H_{G'}$ (Figure 8).

Since H' is a Hamiltonian cycle that entirely consists of inter-clique edges and $|V'| > B_\kappa$, the hypothesis implies that there exists another Hamiltonian cycle of G' that contains strictly less inter-clique edges than H'. Therefore, there exists a Hamiltonian cycle of G that contains less inter-clique edges than H. □

Lemma 6 implies that, in order to prove the upper bound on the cardinality of a minimum HC-set, it suffices to prove the same upper bound on the number of vertices of Hamiltonian graphs that (i) can be partitioned into κ cliques, and (ii) only contain Hamiltonian cycles entirely consisting of inter-clique edges. It should be intuitively clear that such graphs cannot contain an arbitrarily large number of vertices.

Lemma 7. *Given a graph $G(V, E)$ and a partition of V into κ cliques, G is Hamiltonian iff there exists an HC-set M such that $|M| \leq \kappa(\kappa - 1)$.*

Proof. By definition, the existence of an HC-set implies that G is Hamiltonian. Conversely, let H be the Hamiltonian cycle of G that contains the minimum number of inter-clique edges, and let M be the corresponding HC-set. If $|M| \leq \kappa(\kappa-1)$, then we are done. Otherwise, Lemma 6 implies that it suffices to prove the same upper bound on $|V|$ for graphs $G(V,E)$ that only contain Hamiltonian cycles entirely consisting of inter-clique edges.

Assume that $|V| = |M|$ and the coloring of V under M entirely consists of RED vertices, and consider an arbitrary orientation of the Hamiltonian cycle H (e.g. a traversal of the edges of H in the clockwise direction). If there exist a pair of cliques C_i and C_j and four vertices $v_1, v_2 \in C_i$ and $u_1, u_2 \in C_j$, such that both v_x are followed by u_x $(x = 1, 2)$ in a traversal of H, then the BLACK edges v_1v_2 and u_1u_2 can be used instead of v_1u_1 and v_2u_2 in order to obtain a Hamiltonian cycle containing less inter-clique edges than H. The previous situation can be avoided only if, for all $i = 1, \ldots, \kappa$, and $j = 1, \ldots, \kappa$, $j \neq i$, at most one vertex $v_j \in C_i$ is followed by a vertex $u \in C_j$ in any traversal of H. Hence, if $|V| > \kappa(\kappa-1)$, then G contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges. Alternatively, any HC-set M of minimum cardinality contains at most two edges between any pair of cliques C_i and C_j. Thus, M contains at most $\kappa(\kappa-1)$ inter-clique edges. □

The main result of this section is an immediate consequence of the previous lemmas.

Theorem 6. *Given a graph $G(V,E)$, $|V| = n$, and a partition of V into $\kappa > 1$ cliques, there exists a deterministic algorithm that runs in time $\mathcal{O}\big(n^{\kappa(2\kappa-1)}\big)$ and decides if G is Hamiltonian. If G is Hamiltonian, the algorithm outputs a Hamiltonian cycle.*

Proof. We can decide if G is Hamiltonian in time $\mathcal{O}\big(n^{\kappa(2\kappa-1)}\big)$, because the number of the different edge sets containing at most $\kappa(\kappa-1)$ inter-clique edges is at most $n^{2\kappa(\kappa-1)}$, and we can decide if a set of inter-clique edges is an HC-set in time $\text{poly}(n,\kappa) = \mathcal{O}(n^{\kappa})$. □

A BLUE Hamiltonian cycle is a Hamiltonian cycle that does not contain any RED vertices or edges. We can substantially improve the bound of $\kappa(\kappa-1)$ for graphs containing a BLUE Hamiltonian cycle.

Lemma 8. *Given a graph $G(V,E)$ and a partition of V into κ cliques, if G contains a* BLUE *Hamiltonian cycle, then there exists an HC-set M entirely consisting of* BLUE *edges, such that $|M| \leq 2(\kappa-1)$.*

Proof. Let H be the BLUE Hamiltonian cycle of G that contains the minimum number of inter-clique edges and let M be the corresponding HC-set. Assume that $|M| > 2(\kappa-1)$, otherwise we are done. Notice that, since RED vertices cannot be created by removing edges, any $M' \subseteq M$ that corresponds to an Eulerian, connected, clique graph $T(C, M')$ is an HC-set that only contains BLUE edges.

Let $S_T(C, M_S)$ be any spanning tree of $T(C, M)$. Since $|M_S| = \kappa - 1$, the graph $T^{(S)}(C, M \backslash M_S)$, which is obtained by removing the edges of the spanning tree from T, contains at least κ edges. Therefore, $T^{(S)}$ contains a simple cycle L. The removal of the edges of L does not affect connectivity (since the edges of L do not touch the spanning tree S_T), and reduces the degree of each involved vertex by 2. Clearly, the clique graph $T'(C, M \backslash L)$ is connected and Eulerian, and $M \backslash L$, $|M \backslash L| < |M|$, is an HC-set. □

There exist HC-sets of cardinality exactly $2(\kappa-1)$ that correspond to a Hamiltonian cycle using the minimum number of inter-clique edges. Therefore, the bound of $2(\kappa-1)$ is tight. However, we are not able to construct HC-sets that contain more than $2(\kappa-1)$ edges and correspond to Hamiltonian cycles using the minimum number of inter-clique edges. Hence, we conjecture that the bound of $2(\kappa-1)$ holds for any graph and any partition into κ cliques. An inductive (on $|V|$) application of Lemma 6 suggests that this conjecture is equivalent to the following:

Conjecture 1. For any Hamiltonian graph $G(V,E)$ of $2\kappa - 1$ vertices and any partition of V into κ cliques, there exists at least one Hamiltonian cycle not entirely consisting of inter-clique edges.

Remark. Notice that our algorithm works for all graphs, integers $\kappa > 1$, and partitions into κ cliques. Also, any graph can be partitioned into a (non-constant) number of cliques in polynomial time (e.g. maximum matching). Hence, any (deterministic) algorithm that (i) exploits a partition into κ cliques for deciding Hamiltonicity, and (ii) works for all graphs, integers $\kappa > 1$, and partitions into κ cliques (as our algorithm does), should run in time exponential in κ (e.g. $n^{\mathcal{O}(\kappa)}$, $2^{\mathcal{O}(\kappa)}$), unless $\mathcal{NP}$ has (deterministic) subexponential simulations.

A Reduction from Radio Labelling to Hamiltonian cycle

Lemma 9. *Given a graph $G(V,E)$, with $|V| = n$, and a coloring of G with κ colors, an optimal radio labelling of G can be computed in $\mathcal{O}(n^{\kappa(2\kappa+1)})$ time.*

Proof. Let $\overline{G}$ be the complement of the input graph G, and let $E(\overline{G}) = \overline{E} = V \times V \backslash E$. Clearly, $\lambda_2(G) \leq n + \kappa - 1$. Therefore, at most $\kappa - 1$ channels remain unused by an optimal radio labelling. Thus, any optimal solution to the HP(1,2) instance that corresponds to $\overline{G}$ (see also the proof of Lemma 4) contains at most $\kappa - 1$ non-edges (of $\overline{G}$), while any optimal tour (a simple spanning cycle) contains at most κ non-edges (of $\overline{G}$). Then, we show how to compute an optimal TSP(1,2) solution to the complementary graph $\overline{G}$. Clearly, we can delete an edge of maximum length from the optimal tour so as to obtain an optimal spanning path.

Let $\mathcal{A}$ be the algorithm of Theorem 6. We call $\mathcal{A}$ at most $M = \mathcal{O}(n^{2\kappa})$ times, with input the graphs $\overline{G_i}(V, \overline{E} \cup N_i)$, $i = 1, \dots, M$. The sets N_i are all possible subsets of non-edges of $\overline{G}$ with at most κ elements, including the empty one. Let $\overline{G_i}$ be a Hamiltonian graph that corresponds to a set N_i of minimum cardinality. Clearly, the Hamiltonian cycle produced by $\mathcal{A}(\overline{G_i})$ is an optimal tour for $\overline{G}$. □

Since any planar graph can be colored with a constant number of colors in polynomial time, the following theorem is an immediate consequence of Lemma 9.

Theorem 7. *An optimal radio labelling of a planar graph can be computed in polynomial time.*

Remarks. Theorem 7 suggests that an approximation algorithm for 2-coloring a planar graph may be obtained as follows: At first, (partially) decompose the graph to subgraphs, so that almost all vertices of each subgraph get distinct colors by an optimal (or near optimal) assignment; Then, compute an optimal radio labelling for these (planar) subgraphs; Finally, combine the partial assignments to obtain a near optimal assignment.

Clearly, Conjecture 1 implies that, given a graph $G(V,E)$ and a coloring of V with κ colors, optimal solutions for radio labelling can be computed in time $n^{\mathcal{O}(\kappa)}$. □

4.4 An $\mathcal{NC}$ Approximation Algorithm for Arbitrary Graphs

Next, we present an $\mathcal{NC}$ algorithm which, for any graph $G(V,E)$, produces a radio labelling of value at most $|V| + \text{MC}_G - 1$, and a clique of cardinality at least $\lambda_2(G) - |V| + 1$. Since $\lambda_2(G) \geq 2\text{MC}_G$, our algorithm achieves in $\mathcal{NC}$ an approximation guarantee of $\frac{3}{2}$ for arbitrary graphs.

If we consider graphs with small chromatic number (e.g. constant or poly-logarithmic), then our $\mathcal{NC}$ algorithm always produces a radio labelling within a small additive term of the optimal, without assuming a near optimal coloring or any knowledge on the actual chromatic number of the underlying graph. In this sense, the $\mathcal{NC}$ approximation algorithm complements the exact algorithm presented in the previous section, which assumes a near optimal coloring.

As before, for clarity of presentation, the $\mathcal{NC}$ approximation algorithm is described and analyzed in the context of Hamiltonian Paths and Cycles. In particular, we obtain the first $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for HP(1,2) and TSP(1,2). Up to now, the best known parallel algorithm for TSP(1,2)

```
Algorithm TSP-MIS
Input:  A graph G(V,E).
Output: A set P of edges that cover V with vertex disjoint simple paths.
        A set S* of independent vertices.

     P := ∅;
     i := 0;
     repeat
         i := i + 1;    /* i^th phase */
         V^i := {v ∈ V : deg_P(v) = 0};
         For any non-trivial path p ∈ P,
             add exactly one of the end vertices of p to V^i;
         E^i := {{v,w} ∈ E : v ∈ V^i ∧ w ∈ V^i};
         Find a Maximal Matching M^i in G^i(V^i, E^i);
(s1)     P := P ∪ M^i;
         S^i := {v ∈ V^i : deg_{M^i}(v) = 0};
     until M^i is empty or i > K;
     S* := S^i of maximum cardinality;
     return(P, S*);
```

Fig. 9. The Algorithm TSP-MIS.

and all the special cases of metric TSP is the algorithm of Christofides [19], which is based on the computation of a minimum weighted matching (maximum cardinality matching for TSP(1,2)); this is in $\mathcal{RNC}$, but it is not known to be in $\mathcal{NC}$.

The Algorithm for HP(1,2) and TSP(1,2) Let $G(V,E)$ be any graph of n vertices, and TSP_G be the optimal value of the corresponding TSP(1,2) instance.

Theorem 8. *The following inequalities hold for any graph $G(V,E)$ of n vertices:*

$$2\mathrm{MIS}_G \leq \mathrm{TSP}_G \leq n + \mathrm{MIS}_G \tag{1}$$

Moreover, a tour and an independent set that fulfil (1) can be computed in $\mathcal{NC}$.

Proof. The main part of the proof consists of the algorithm TSP-MIS (Figure 9) that computes a tour and an independent set of G that fulfil (1).

Correctness: The algorithm proceeds in phases. During all phases, it maintains a set P of simple paths (initially, $P = \emptyset$). In phase i, the algorithm computes a Maximal Matching M^i in the subgraph $G^i(V^i, E^i)$ induced by the vertex set V^i, which consists of exactly one of the end vertices of all the paths included in P. The algorithm adds the edges of M^i to P, and proceeds to the next phase. Notice that since any vertex v of V^i has $\deg_P(v) \leq 1$, P remains a collection of simple paths even after the inclusion of the edges of M^i.

Therefore, at the end of the algorithm, the set P consists of simple paths (of 1-edges) and isolated vertices. If the isolated vertices are considered as paths of length 0 (trivial paths), the edge set P covers V with vertex disjoint simple paths. Additionally, the vertex sets S^i are independent sets, because M^i is a maximal matching in $G^i(V^i, E^i)$.

Performance: The performance of the algorithm is determined by the number of edges (of length 1) that are contained in the set P at the end of the algorithm. Initially, the set P is empty. At each phase i, the edges of M^i are added to P in step (s1). Let $|M^i|$ be the number of edges of a maximal matching M^i of the graph G^i. The algorithm runs for $K+1$ phases, i.e. $i = 1, \ldots, K+1$, where K will be determined later. Clearly, $|P| = \sum_{i=1}^{K} |M^i|$.

Since the vertices of V^i that are not covered by a maximal matching M^i form an independent set, we obtain that

$$\frac{|V^i| - \text{MIS}_G}{2} \leq |M^i| \leq \frac{|V^i|}{2}$$

Furthermore, for any pair of vertices of V^i that are matched by M^i, exactly one vertex is added to V^{i+1}. Therefore,

$$\begin{aligned} \frac{|V^i|}{2} &\leq |V^{i+1}| \\ &= |V^i| - |M^i| \\ &\leq \frac{|V^i| + \text{MIS}_G}{2} \end{aligned}$$

The following inequality holds for all $i \geq 1$, and can be proved by induction on i.

$$|V^{i+1}| \leq \frac{|V^1| + (2^i - 1)\text{MIS}_G}{2^i} \tag{2}$$

Equation (2) holds for $i = 1$, because

$$|V^2| \leq \frac{|V^1| + \text{MIS}_G}{2} .$$

Inductively, we assume that it holds for some $i \geq 1$, and we show that it also holds for $i + 1$. The previous inequalities imply that

$$\begin{aligned} |V^{i+2}| &\leq \frac{|V^{i+1}| + \text{MIS}_G}{2} \\ &\leq \frac{\frac{|V^1| + (2^i - 1)\text{MIS}_G}{2^i} + \text{MIS}_G}{2} \\ &= \frac{|V^1| + (2^{i+1} - 1)\text{MIS}_G}{2^{i+1}} \quad , \end{aligned}$$

where the second inequality holds because of inductive hypothesis.

If $K = \lceil \log n \rceil$, then inequality (2) implies that

$$|V^{K+1}| \leq \frac{|V^1| + (n-1)\text{MIS}_G}{n} < \text{MIS}_G + 1$$

By summing the equalities $|V^{i+1}| = |V^i| - |M^i|$, for $1 \leq i \leq K+1$, we obtain by sum telescoping that

$$|V^{K+1}| = |V^1| - \sum_{i=1}^{K} |M^i| = |V| - |P|$$

Therefore, we obtain that $|P| \geq n - \text{MIS}_G$, because if $K = \lceil \log n \rceil$, then $|V^{K+1}| \leq \text{MIS}_G$. Since the set P consists of simple paths of total length at least $n - \text{MIS}_G$, we can construct a spanning cycle from P by adding no more than MIS_G edges of length 2. The length of the resulting tour will be no more than $n + \text{MIS}_G$. Therefore, $\text{TSP}_G \leq n + \text{MIS}_G$.

Complexity: The algorithm runs for at most $\lceil \log n \rceil + 1$ phases. Since the connected components of P are simple paths, the computation of G^i can be implemented in $\mathcal{NC}$. The complexity of each phase of the algorithm is dominated by the computation of the maximal matching. There exist a CRCW PRAM algorithm that produces a maximal matching in $\mathcal{O}(\log^3 n)$ time using $\mathcal{O}(n^2)$ processors [52]. Moreover, the sequential complexity of TSP-MIS is $\mathcal{O}(n^2 \log n)$, because each phase can be implemented in sequential time $\mathcal{O}(n^2)$.

Next, we prove that $2\text{MIS}_G \leq \text{TSP}_G$. Let $\text{TSP}_G = n + \gamma$, for any $0 \leq \gamma \leq n$, and let $\pi : [n] \mapsto V$ be a permutation that corresponds to an optimal tour of G. Clearly, π defines a set $\{p_1, p_2, \ldots, p_\gamma\}$ of

γ simple paths, which consist of 1-edges and are linked to a spanning cycle by 2-edges. Clearly, if a path p_i consists of $|p_i|$ vertices, it cannot contribute to any independent set more than $\frac{|p_i|+1}{2}$ vertices, if $|p_i|$ is odd, and $\frac{|p_i|}{2}$ vertices, if $|p_i|$ is even. Therefore, the cardinality of any independent set in G cannot be greater than the sum of the cardinalities of the independent sets of p_i's. Consequently,

$$\begin{aligned} \mathrm{MIS}_G &\leq \sum_{i=1}^{\gamma} \left\lceil \frac{|p_i|}{2} \right\rceil \\ &\leq \sum_{i=1}^{\gamma} \frac{|p_i|+1}{2} \\ &\leq \frac{n+\gamma}{2} = \frac{\mathrm{TSP}_G}{2} \end{aligned}$$

It remains to prove that the algorithm TSP-MIS computes in $\mathcal{NC}$ an independent set of cardinality at least $\mathrm{TSP}_G - n$. Assume that $\mathrm{TSP}_G - n = \gamma > 0$. We run TSP-MIS for $K = \lceil \log n \rceil$ phases, and we output the set $S^\star$, namely the set S^i, $i = 1, \ldots, K+1$, of maximum cardinality. Clearly, S^i's are sets of independent vertices. If $|S^\star| = \gamma' < \gamma$, we can substitute the MIS_G with γ' in the analysis on the size of P. Therefore, if $|S^i| < \gamma$, for all $1 \leq i \leq K+1$, the resulting set P would contain more than $n - \gamma$ edges. This is a contradiction, because it implies that we can compute a tour better than the optimal. Consequently, the algorithm TSP-MIS always produces an independent set of cardinality at least $\mathrm{TSP}_G - n$.

There exist instances such that $\mathrm{TSP}_G = n + \mathrm{MIS}_G - 1$. For example, for any $x > 1$, consider a graph G consisting of a clique of $n - x$ vertices and an independent set of x vertices, all of them connected to the same vertex of the clique. Clearly, $\mathrm{MIS}_G = x + 1$, and $\mathrm{TSP}_G = n + x$.

Also, the inequality $2\mathrm{MIS}_G \leq \mathrm{TSP}_G$ becomes tight for many instances. It is worth mentioning the C_n graph (a simple cycle of n vertices), and the complete bipartite graph, where the sizes of the classes are δn and $(1-\delta)n$, for any $\delta > 0$. □

Some Consequences of Theorem 8 Clearly, the arguments above also apply to HP(1,2) and to radio labelling for the complementary graph $\overline{G}$. Therefore, the following are immediate consequences of Theorem 8:

Corollary 2. *There exists an $\mathcal{NC}$ algorithm that runs in a CRCW PRAM in time $\mathcal{O}(\log^4 n)$ using $\mathcal{O}(n^2)$ processors and approximates,*

1. *Radio labelling, HP(1,2), and TSP(1,2) within $\frac{3}{2}$.*
2. *HP(1,2) and TSP(1,2) restricted to graphs G, such that $\mathrm{MIS}_G \leq \alpha n$, within $(1+\alpha)$.*
3. *Radio labelling restricted to graphs G, such that $\mathrm{MC}_G \leq \alpha n$ ($\chi(G) \leq \alpha n$), within $(1+\alpha)$.*
4. *Maximum Clique restricted to graphs G, such that $\lambda_2(G) \geq (1+\beta)n$, within $\frac{2\beta}{1+\beta}$.*

4.5 Graphs with Bounded Maximum Degree

Although radio labeling is better approximable in graphs with bounded chromatic and clique numbers, we show that this is not the case for graphs with bounded maximum degree. In particular, we prove that there exists a constant $\Delta^\star < 1$, such that for all $\gamma \in [\Delta^\star, 1)$, γ-bounded instances of radio labeling (i.e. $\Delta(G) \in [\Delta^\star|V|, |V|)$) are essentially as hard to approximate as general instances. On the other hand, an optimal radio labeling of value $|V|$ can be found in polynomial time for any graph $G(V,E)$ with maximum degree less than $\frac{|V|}{2}$.

Theorem 9. *There exist a constant $\Delta^\star < 1$, such that, for all γ such that $1 > \gamma \geq \Delta^\star$, γ-bounded instances of radio labeling do not admit a PTAS, unless $\mathcal{P} = \mathcal{NP}$.*

Proof. Let $G(V, E)$ be any graph of n vertices. Clearly, $n \leq \lambda_2(G) \leq 2n$. For any constant γ such that $1 > \gamma > \frac{1}{2}$, let $\delta = \frac{1-\gamma}{\gamma}$, and consider $K_{\delta n}$, the complete graph on δn vertices. We construct a graph G' by adding $K_{\delta n}$ to G. The vertices of G are not connected to the vertices of $K_{\delta n}$. Formally, the new graph is $G'(V \cup K_{\delta n}, E')$, where

$$E' = E \cup \{uv : u, v \in K_{\delta n}\}$$

Obviously, since $\delta = \frac{1-\gamma}{\gamma}$, the maximum degree of any vertex of G' is bounded by γn.

By the construction of G', if we are given any radio labeling $L_{G'}$ of G', we can find a radio labeling L_G of G of value $\lambda_2(L_G) \leq \lambda_2(L_{G'})$. (Ignore the labels assigned to the vertices of $K_{\delta n}$ in $L_{G'}$. This always results in a radio labeling of G.)

Conversely, given any radio labeling L_G of G, we can construct a radio labeling $L_{G'}$ of G', such that the vertex arrangements implied by the labelings $L_{G'}$ and L_G are the same with respect to the vertices of G. Moreover, the value of the labeling $L_{G'}$ is

$$\lambda_2(L_{G'}) = n + \max(\delta n, n - \lambda_2(L_G))$$

Let λ be a label not assigned by L_G, and let $u, v \in V$ be the vertices assigned the labels $\lambda - 1$ and $\lambda + 1$, i.e. $L_G(u) = \lambda - 1$ and $L_G(v) = \lambda + 1$. Given a label not assigned by L_G, such a vertex pair exists, because we only consider labelings of minimal value. Then, an arbitrary vertex of $K_{\delta n}$ is assigned the label λ. Clearly, if the labels not assigned by L_G are at most δn, then all the labels up to $(1+\delta)n$ will be assigned by $L_{G'}$. Consequently, $\lambda_2(L_{G'}) = (1+\delta)n$. Otherwise ($\lambda_2(L_G) \geq (1+\delta)n$), the labels assigned to $K_{\delta n}$ cannot increase the value of the resulting radio labeling. Therefore, $\lambda_2(L_{G'}) = \lambda_2(L_G)$. Therefore, the optimal values $\lambda_2(G)$ and $\lambda_2(G')$ fulfil the following inequalities:

$$\lambda_2(G') - \delta n \leq \lambda_2(G) \leq \lambda_2(G') \leq (1+\delta)\lambda_2(G)$$

Assume that for some constant $\epsilon > 0$, radio labeling is approximable within $(1+\epsilon)$ for γ-bounded graphs. Given any graph G, we can construct a γ-bounded graph G' by adding a clique on δn vertices, $\delta = \frac{1-\gamma}{\gamma}$, and find an $(1+\epsilon)$ approximate solution $L_{G'}$ to this instance. Then, we can transform $L_{G'}$ to a radio labeling L_G for the original graph G by ignoring the labels assigned to the additional vertices. Clearly, the following inequalities hold:

$$\lambda_2(L_G) \leq \lambda_2(L_{G'}) \leq (1+\epsilon)\lambda_2(G') \leq (1+\epsilon)(1+\delta)\lambda_2(G)$$

Since radio labeling is MAX–SNP-hard, there exists a constant $\epsilon^\star > 0$, such that radio labeling is not approximable within $(1+\epsilon^\star)$, unless $\mathcal{P} = \mathcal{NP}$ [9]. Let $\epsilon^\star > \alpha > 0$ be any small constant, and $\Delta^\star = \frac{1+\alpha}{1+\epsilon^\star} < 1$. If γ-bounded instances of radio labeling admitted a PTAS for some $\gamma \geq \Delta^\star$, then general instances would be approximable within $(1+\epsilon^\star)$.

It is straightforward to verify that the same arguments can be applied even if a small constant number of edges are placed between the vertices of G and the vertices of $K_{\delta n}$ so as G' to be a connected graph. □

Moreover, the previous transformation implies that approximating radio labeling in γ-bounded instances is essentially as hard as approximating radio labeling in general instances. This becomes clear by considering instances with large optimal values, namely graphs G such that $\lambda_2(G) \geq \left(1 + \frac{1-\gamma}{\gamma}\right) n$, for some constant $1 > \gamma \geq \Delta^\star$. Then, the previous transformation is an L-reduction (e.g., see [80]) from such instances of radio labeling to γ-bounded instances.

A similar result from HP(1,2) and TSP(1,2) follows from the fact that radio labeling is equivalent to HP(1,2) in the complementary graph.

Corollary 3. *There exist a constant $\delta^\star > 0$, such that, for all $0 < \delta \leq \delta^\star$, δ-dense instances of HP(1,2) and TSP(1,2) do not admit a PTAS, unless $\mathcal{P} = \mathcal{NP}$.*

5 Radiocoloring

5.1 Introduction, Previous Work and Our Results

The problem of frequency assignment in radio networks is a well-studied one. The interference among transmitters is modeled by an interference graph $G(V, E)$, where V ($|V| = n$) corresponds to the set of transmitters and E represents distance constraints (e.g., if two neighbor nodes in G get the same or close frequencies, then this causes unacceptable levels of interference). In most real life cases the network topology has some special properties; e.g., G is a lattice network or a planar graph. Planar graphs are the object of study in this work.

The Frequency Assignment Problem (FAP) is usually modeled by variations of the graph coloring problem. The set of colors represents the available frequencies. In addition, each color in a particular assignment gets an integer value, which has to satisfy certain inequalities compared to the values of colors of nearby nodes in G (frequency-distance constraints). The FAP has been considered in e.g. [40, 28, 29]. Despite the important work done in either lattices or general networks, almost nothing has been reported for planar interference graphs, with the exception of [8].

In the sequel, let $d(u, v)$ be the minimum distance of u, v in G. A discrete version of FAP is the k-coloring problem:

Definition 16. k-coloring*: Given a graph $G(V, E)$ consider a function $\Phi : V \rightarrow \{1, \ldots, \infty\}$ such that $\forall\ u, v \in V$ $|\Phi_u - \Phi_v| \geq x$ for $x = 0, 1, \ldots, k$ only if $d(u, v) \geq k - x + 1$. This function is called a k-coloring of G. Let $|\Phi(V)| = \lambda$. Then λ is the* number of colors *that Φ actually uses (it is usually called* the color order *of G). The number $\nu = max_{v \in V}\Phi(v) - min_{u \in V}\Phi(u) + 1$ is usually called the* span *of G.*

Note that the case $k = 1$ corresponds to the well known problem of vertex graph coloring. Thus, k-coloring (with k as an input) is $\mathcal{NP}$-complete. More specifically, we are interested in a variation of the k-coloring problem, where $k = 2$:

Definition 17. Radiocoloring Problem *(RCP): Given a graph $G(V, E)$ consider a function Φ : $V \rightarrow N^*$ such that $|\Phi(u) - \Phi(v)| \geq 2$ if $d(u, v) = 1$ and $|\Phi(u) - \Phi(v)| \geq 1$ if $d(u, v) = 2$. The least possible number of colors (order) that can be used to radiocolor G is denoted by $X_{order}(G)$. The number $\nu = max_{v \in V}\Phi(v) - min_{u \in V}\Phi(u) + 1$ is called span of the radiocoloring of G and the least such number is denoted as $X_{span}(G)$.*

Real networks reserve bandwidth (range of frequencies) rather than distinct frequencies. It is sometimes desirable to use as less distinct frequencies of a given bandwidth (span) as possible, since the unused frequencies are available for other uses. This optimization version of the Radiocoloring Problem (RCP) is studied in this work and is defined as follows.

Definition 18. Min span order RCP*: The optimization version of Radiocoloring that minimizes the span is called* min span RCP*. The* min span order RCP *from all minimum span assignments tries to find one that uses as few colors as possible. The optimal span is called X_{span}. The order of the computed assignment is called X'_{order}.*

However, there are cases where the primary objective is the number of colors used but the span is a secondary objective since we do not need to reserve unnecessary large span. This problem is defined as follows:

Definition 19. Min order span RCP*: The optimization version of Radiocoloring that minimizes the span is called* min span RCP*. The* min span order RCP *from all minimum span assignments tries to find one that uses the minimum number of colors, (min order). The optimal span is called X_{span}. The minimum order of the computed assignment is called X'_{order}.*

It easy to see that $X_{order} \leq X'_{order}$ and $X_{span} \leq X'_{span}$. Also, it holds that $X_{order} \leq X_{span}$. Another variation of FAP is related to the square of a graph G defined as follows:

Definition 20. *Given a graph $G(V, E)$, G^2 is the graph of the same vertex set V and an edge set $E' : uv \in E'$ iff $d(u, v) \leq 2$ in G.*

The variation of FAP addressed here is to color G^2 with the minimum number of colors, denoted as $\chi(G^2)$.

Claim. For any graph G, $X_{order}(G)$, is the same as the (vertex) chromatic number of G^2, i.e. $X_{order}(G) = \chi(G^2)$.

Proof. Assume, in contrary, that $X(G^2) < X_{order}(G)$. Then from the optimal coloring of G^2 we can obtain a radiocoloring of G with $\chi(G^2)$ colors by doubling the assigned color of each node. In this way, we get a new radio coloring assignment of G with less than $X_{order}(G)$ colors, which contradicts the definition of $X_{order}(G)$. Assume now $\chi(G^2) > X_{order}(G)$. From the optimal min order radiocoloring, we can easily get a coloring of G^2 assigning to each node the same color as in the radiocoloring assignment. The assignment is valid for the coloring of G^2, since both distance 1 and 2 constraints hold in any radiocoloring. Thus, we find a new coloring of G^2 with less than $\chi(G^2)$ colors, which contradicts the definition of $\chi(G^2)$. Thus, $\chi(G^2) = X_{order}(G)$.

However, notice that although the number of colors used in a minimal coloring of G^2 and a min order span radiocoloring is the same, the set of colors in the two solutions need not be the same. To see this, recall the previous argument showing that from an optimal coloring of G^2, we can obtain an optimal min order span radiocoloring by doubling the assigned color to each node. □

Observe also that $\chi(G^2) \leq X_{span} \leq 2\chi(G^2)$. It is obvious that $\chi(G^2) \leq X_{span}$. Furthermore, notice that from a coloring of G^2 we can always obtain a radiocoloring of G by multiplying the assigned color of every vertex by two. The resulting radiocoloring has span $2\chi(G^2)$.

In [40] and [28], it has been proved that the problem of minimizing the span in radiocoloring is $\mathcal{NP}$-complete, even for graphs of diameter 2. The reductions use highly non-planar graphs. In [67], it is proved that the problem of coloring the square of a general graph is $\mathcal{NP}$-complete.

In [8] a similar problem for *planar* graphs has been considered. This is the *Hidden Terminal Interference Avoidance (HTIA)* problem, which requests to color the vertices of a planar graph G so that vertices *at minimum distance exactly 2* get different colors. In [8], this problem is shown to be $\mathcal{NP}$-complete.

However, the above mentioned result does not imply the $\mathcal{NP}$-hardness of the radiocoloring problem. This is so because in HTIA it is allowed to color neighbors in G with the same color. The minimum number of colors thus needed for HTIA can vary arbitrary from the $X_{order}(G)$. To see this consider the *t-size* clique graph K_t. In HTIA, this can be colored with only one color. In our case (RCP) we need t colors for K_t. In addition, the reduction used in [8] (from the coloring problem of straight line planar graphs of degree at most 3), heavily exploits the fact that neighbors in G get the same color in the component substitution part of the reduction. Thus, their reduction cannot be easily modified to produce an $\mathcal{NP}$-hardness proof for RCP.

Thus, any polynomial time decision procedure for RCP does not imply a decision procedure for HTIA in the case it yields a "No" answer. Also, any polynomial time decision procedure for HTIA does not give a decision for RCP in the case of "Yes" answer. In fact, the minimum number of colors needed for HTIA is the chromatic number of $G^2 \backslash G$. The relation between $\chi(G^2)$ and $\chi(G^2 \backslash G)$ for a planar G is not known.

Another variation of FAP for planar graphs, called *distance-2-coloring* is studied in [84]. This is the problem of coloring a given graph G with the minimum number of colors so that the vertices of distance at most two get different colors. Note that this problem is equal to coloring the square of the graph

G. In the above work it is proved that the distance-2-coloring for planar graphs is $\mathcal{NP}$-complete. As we show, in this work this problem is different that the min span order RCP considered here, which is proved here for the first time to be $\mathcal{NP}$-complete. Thus, the $\mathcal{NP}$-completeness of [84] certainly does not imply the $\mathcal{NP}$-completeness of min span order RCP proved here. Additionally, the $\mathcal{NP}$-completeness proof of this work does not work for planar graphs of maximum degree $\Delta > 7$. Hence, their proof gives no information on the complexity of distance-2-coloring of planar graphs of maximum degree > 7. In construct, our $\mathcal{NP}$-completeness proof works for planar graphs of all maximum degrees. The same work [84] provides a 9-approximation algorithm for the distance-2-coloring of planar graphs.

In this section, we are interested in *min span order*, *min order span* and *min span* in a radiocoloring of a *planar* graph G.

(a) We first show that the minimum number of colors $X'_{order}(G)$ used in the *min span order RCP* of graph G is different from the chromatic number of the square of the graph ($\chi(G^2)$).

(b) We show that *the problems of min span order RCP, min order span and min span are $\mathcal{NP}$-complete* for planar graphs. Note that few combinatorial problems remain hard for *planar* graphs and their proofs of hardness are not easy, since they have to use planar gadgets which are difficult to find and understand [64]. As we argued before, this result is *not* implied by the $\mathcal{NP}$-completeness results of similar problems [8, 84].

(c) We then present an $O(n \max\{\log n, \Delta\})$ algorithm that approximates the minimum number of colors of a radiocoloring, $X_{order}(G)$, of a planar graph G by a constant ratio which tends to 2 as the maximum degree of G increases. Our algorithm is motivated by a constructive proof of a coloring theorem presented by Heuvel and McGuiness [49]. Their construction can easily lead (as we show) to an $O(n^2)$ technique, assuming that a planar embedding of G is given. We improve the time complexity of the approximation, and we present a much more simple algorithm to verify and implement. Our algorithm does not need any planar embedding as input.

(d) Finally, we study the problem of *estimating the number of different radio colorings* of a planar graph G. This is a #P-complete problem (as can be easily seen from our completeness reduction, that can be done parsimonious). We employ here standard techniques of rapidly mixing Markov Chains and the *new method of coupling* for the purpose of proving *rapid convergence* (see, e.g., [50]), and we present *a fully polynomial randomised approximation scheme* for estimating the number of radio colorings with λ colors for a planar graph G, when $\lambda \geq 4\Delta + 50$, where Δ is the maximum vertex degree of G.

5.2 Radiocoloring Versus Distance-2-coloring in Planar and General Graphs

The work of [84] studied a relative to RCP problem, called em distance-2-coloring (and defined in Section 1). It is easily seen that the problem of distance-2-coloring a graph G is equivalent to the problem of coloring G^2. In [84], it is proved that the problem of distance-2-coloring for planar graphs is $\mathcal{NP}$-complete. The following surprising result proves that the min span order RCP has different order than the distance-2-coloring (or the coloring of G^2). Thus, the $\mathcal{NP}$-completeness of distance-2-coloring does not imply the $\mathcal{NP}$-completeness of min span order radiocoloring. Moreover, we remark that the $\mathcal{NP}$-completeness proof of [84] does not work for planar graphs of maximum degree $\Delta \geq 8$, as opposed to our $\mathcal{NP}$-completeness proof for the min span order RCP.

Theorem 10. *There is at least one instance G of the min span order RCP whose solution is different from the solution of the problem of coloring the square of graph G (distance-2-coloring of G).*

Proof. Consider the instance G of the two problems appearing in Figure 10. It is easy to see that the problem of distance-2-coloring is solved optimally with 6 colors, while the span of the min span order RCP is at least 7. The assignment of Figure 10 shows that the span and the order of a radiocoloring assignment of G is at most 8.

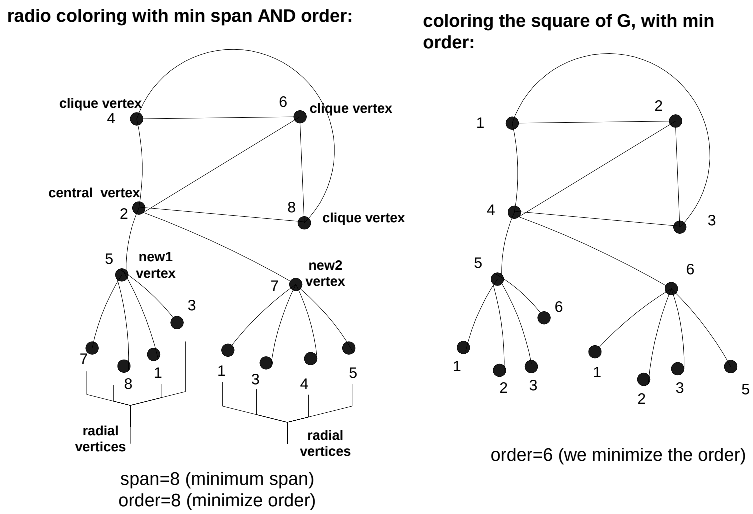


Fig. 10. An instance where the problem of min span order radio coloring and the problem of distance-2-coloring have solutions of different orders.

We assert that the min span order RCP has an order and span at least 8. Assume that this is not true. Then let the order of the optimal solution to be 6. The vertices of the graph are named as shown in the Figure 10. The *endmost colors* of a palette of colors (integers) S from which some or all of its colors are used in an assignment are the colors of the smallest and largest integer in the set S that are used in the assignment. For example in the set $S = \{1, \ldots, 8\}$ the colors $1, 8$ are the two end most colors of set S.

Firstly, observe that the $new1$, $new2$ vertices should take endmost colors. In other case, their *adjacent radial* vertices avoid to take 4 colors instead of 3, necessitating the use of more than 7 colors. But, if we assign the two endmost colors to the $new1$, $new2$ vertices, we force the four vertices of the clique formed to use more than 8 colors. Concluding, in any case, the number of colors used is at least 8. □

The claim can also be experimentally verified by running a program that checks exhaustively all feasible solutions. We implemented this program in C (see [31]) and verified the formal result.

5.3 The $\mathcal{NP}$-Completeness of the Radiocoloring problem

In this section, we show that the decision version of min span radiocoloring remains $\mathcal{NP}$-complete for planar graphs. The decision version of min span radio coloring is, given a planar graph G and an integer B, to decide whether there exists a radiocoloring for G of span no more than B. Therefore, the optimization version of min span radiocoloring, that is to compute a radiocoloring of minimum span, remains $\mathcal{NP}$-hard for planar graphs.

Theorem 11. *The min span radiocoloring problem is $\mathcal{NP}$-complete for planar graphs.*

Proof. The decision version of min span radio coloring clearly belongs to the class $\mathcal{NP}$. To prove the theorem, we transform PLANAR-3-COLORING to min span radiocoloring. The PLANAR-3-COLORING problem for a given a planar graph $G(V, E)$ asks if there is a coloring of G using 3 colors, i.e. all vertices of G are assigned a color from a palette of size three so that no two neighbor vertices get the same color. We consider a plane embedding of G.

We construct a graph G' as following: Let F be the set of the faces of G, and let Δ_G be the maximum degree of vertices of G. If $f \in F$ then $size(f)$ is the number of edges of f. Then we define an integer Γ as $\Gamma = \max\{\Delta_G, \max_{i \in F}\{size(i)\}\}$ The term $degree_H(u)$ denotes the degree of vertex u in H. The graph G' is constructed as following. First, we define some of the vertices of G':

(a) *Set* V_1*:* For each vertex v of V, add a new vertex v in G'; hence $V_1 \subseteq V$. Call such a vertex *existing vertex.*

(b) *Set* I*:* For each edge of the set E of G, add a new vertex i in G'; hence $|I| = |E|$. Call such a vertex *intermediate vertex.*

(c) *Set* C*:* For each face of the graph G add a new vertex c. Call such a vertex *central vertex.*

(d) *Set* EH*:* For each vertex $u_i \in V$, add $\Gamma - degree_G(u_i)$ new vertices, where $degree_G(u_i)$ is the degree of the vertex u_i in G. Call such vertices *existing-vertex-hanging-vertices.*

(e) *Set* $EH5$*:* For each vertex $u_i \in V$, add a new vertex. Call such a vertex *existing-vertex-hanging5-vertex.*

(f) *Set* $CH2$*:* For each new central vertex $c_i \in C$ add two new vertices. Call such vertices *central-hanging2-vertices.*

We proceed with the definition of some of the edges of G', in order to be able to complete the definition of the vertices of G'.

(1) *Sets* I, V_1*:* Connect each intermediate vertex of set I with the two existing vertices substituting the edge from which the intermediate vertex is derived, by rule (b). Note that by now, every existing vertex $u_i \in V_1$ has $degree_{G'}(u_i) = degree_G(u_i)$.

(2) *Sets* C, I: Connect each intermediate vertex of set I with one of the central vertices of the faces where the edge from which the intermediate vertex is derived, belongs. If after applying this rule, there is a central vertex $c_i \in C$, for which no intermediate vertex is connected to it, this vertex is isolated, so that it can be ignored in the radiocoloring of G'.

Now, we define the rest of the vertices of G'.

(g) *Set* CH*:* For each central-vertex c_i of set C, related to a face i of set F, we add $\Gamma - |I_i|$ new vertices, where I_i is the set of intermediate vertices of the same face i, connected to c_i by rule (2) above. Call such vertices *central-hanging vertices.*

Finally, we define the rest of the edges of G':

(3) *Sets* EH, V_1: Connect each vertex $u_i \in V$ to the $\Gamma - degree_G(u_i)$ new vertices (existing-vertex-hanging-vertices) added for this vertex by rule (d) above. Note that, this rule (combined with rule (1)) results to a $degree_{G'}(u_i) = \Gamma$ for every vertex $u_i \in V_1$ of G'.

(4) *Sets* $EH5, V_1$: Connect each vertex $u_i \in V$ to the new vertex (existing-vertex-hanging5-vertices) added for this vertex in G' by rule (e). Now every vertex $u_i \in V_1$ has $degree_{G'}(u_i) = \Gamma + 1$.

(5) *Set* $CH2, C$: Connect each central vertex $c_i \in C$ (rule (c)) with the two new vertices (central-hanging2-vertices) added for this vertex in G' by rule (f).

(6) *Set* CH, C: Connect each central vertex $c_i \in C$ with the $\Gamma - |C_i|$ new vertices (central-hanging-vertices) related with c_i by rule (g). Note that, combining this rule with rules (2), (5), we conclude that each central vertex $c_i \in C$ has $degree_{G'}(c_i) = \Gamma + 2$.

Observe that G' is a planar graph. To see why consider the edge set of G'. Each edge of G is subdivided in G'. Obviously, this rule respects the planarity of the initial graph G. Moreover, for each existing vertex we add a number of vertices and connect them only to that vertex. Again, this rule respects the planarity of the initial graph. Also, for each face of G we add a vertex, called central, and connect it to some of the intermediate vertices corresponding to the neighbor to that face, edges. Since the edges are added inside the face, and they form a star, the planarity of the initial graph remains. Finally, for each central vertex, we add a number of vertices and connect them only to that vertex. This rule completes the edge set of G' and also ensures that the planarity of the initial graph G remains. Hence, the graph G' is a planar graph.

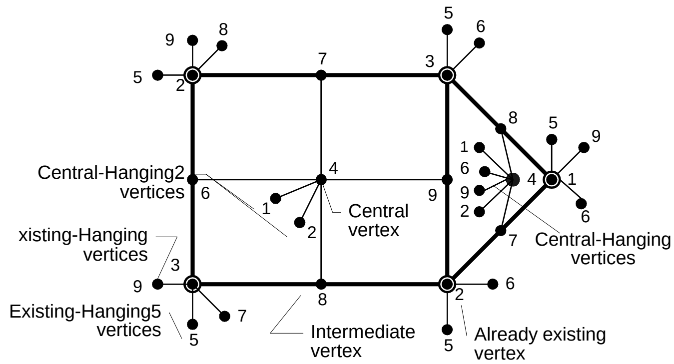


Fig. 11. The graph G' obtained from an instance G of the planar 3-coloring problem (with $\Gamma = 4$).

An example of such a planar construction is presented in figure 11. The edges of graph G are marked with heavy lines and its vertices with bigger cycles.

Consider the following min span order radiocoloring assignment Φ of G': (i) Color all central vertices with color 4. (ii) Color all existing-vertex-hanging5 vertices with color 5. Color the two central-hanging2 vertices of each central vertex with colors 1, 2. (iii) Color the intermediate vertices with colors $6, \ldots, \Gamma+5$. (iv) Color the central-hanging vertices with the unused colors of the set $6, \ldots, \Gamma+5$. (v) Color existing vertices with colors 1, 2, 3. An instance of the suggested assignment is shown in Figure 11.

Claim. If G is 3-colorable, then the suggested radiocoloring assignment Φ of G' has span $\Gamma + 5$.

Proof. Central vertices: Let c any of them. The Γ out of the $\Gamma+2$ neighbors of c (intermediate, central-hanging vertices) get colors $6, \ldots, \Gamma+5$. Also, the two central-hanging2 vertices of it get colors 1 and 2. Since the distance two existing vertices, neighbors of c may get colors 1, 2, 3, it follows that c is allowed to take only colors 4 and 5 from which 4 is at frequency distance two from the colors of its neighbors.

Existing-vertex-hanging5 vertices: Let r be any of them. Vertex r has Γ distance two neighbors (intermediate, existing-hanging vertices) which take in Φ colors $6, \ldots, \Gamma+5$. Also, r is adjacent to an existing vertex which may take one of the colors $1, 2$ and 3. Thus, r can take color 5 without any conflicts with the colors of its distance one and two neighbors.

Intermediate vertices: Let i any of them connected to the existing vertices a, b and a central vertex c. The vertex i is distance two neighbor of a's (distance one) Γ and b's (distance one) Γ neighbors. However, note that a's and b's Γ neighbors are located at distance 3 apart; thus the two sets of vertices can take the same set of colors. Hence, it is sufficient to consider only one of the two sets of Γ vertices. The Γ neighbors of i (in one of the two sets), (rest of intermediate, existing-hanging, existing-hanging5 vertices) get colors $5, \ldots, \Gamma+5$. This set includes $\Gamma+1$ colors, from which color 5 is assigned to the existing-vertex-hanging5 vertex. Hence, in the remaining set of Γ colors $6, \ldots, \Gamma+5$, there will be one color available for i. Note that this color is at frequency distance two from the colors of its existing neighbors as well as from the color of central vertex c (which takes color 4 in Φ).

Existing-hanging vertices: Let r be any of them. Vertex r has Γ distance two neighbors (rest of existing-hanging, intermediate, existing-hanging5 vertices) which take in Φ colors $5, \ldots, \Gamma+5$. This set includes $\Gamma+1$ colors, from which color 5 is assigned to the existing-vertex-hanging5 vertex. Hence, in the remaining set of Γ colors $6, \ldots, \Gamma+5$, there will be one color available for i. Note, that this color differs by at least two from the color assigned to the central vertex, adjacent to it (colored $1, 2$ or 3).

Central-hanging vertices: Let r be any of them. Vertex r has $\Gamma+1$ distance two neighbors (rest of central-hanging, intermediate, central-hanging2 vertices) which take in Φ colors $1, 2$ and $6, \ldots, \Gamma+5$.

These sets include $\Gamma + 2$ colors, from which $1, 2$ are assigned to the two central-hanging2 vertices. Hence, in the remaining set of Γ colors, $6, \ldots, \Gamma + 5$, there will be one color available for r. Note, that this color differs by at least two from color 4 assigned to the central vertex, neighbor to r.

Central-hanging2 vertices: Let r be any of them. Vertex r has $\Gamma + 1$ distance two neighbors (intermediate, central-hanging vertices) which take in Φ colors $1, 2$ and $6, \ldots, \Gamma + 5$. These sets includes $\Gamma + 2$ colors, from which one of the colors $1, 2$, is assigned to the other central-hanging2 vertex. Hence, it is assigned to the existing-vertex-hanging5 vertex. Thus, in the remaining sets of $\Gamma + 1$ colors, one of the colors 1 or 2, is always available for r. Note, that this color differs by at least two from color 4 assigned to the central vertex, neighbor to r.

Existing vertices: Let u be any of them and let c be a central-vertex, which is a distance two neighbor of it. The $\Gamma + 1$ neighbors of u (intermediate, existing-hanging5 vertices) get colors $5, \ldots, \Gamma + 5$. Also, u can not take the color 4 of the central vertex c. Thus, the only constraints it must face are of its distance two existing neighbors. This situation is equivalent to the 3-coloring of G. It follows that G' is radiocolored using a span of size $\Gamma + 5$ if G is 3-colorable.

The above arguments show that a span of size $\Gamma + 5$ is sufficient for the radiocoloring assignment Φ of G' if G is 3-colorable. Furthermore, the last one shows that actually *all* $\Gamma + 5$ colors are used by the assignment.

(End of proof of Claim) □

Thus, we now only need to prove the following claim.

Claim. If there exists a radiocoloring of G' of span $\Gamma + 5$, then G is 3-colorable.

Proof. Assume that G' is radio colored with a span of size $\Gamma + 5$ colors through a radiocoloring assignment Φ_0 of span of the same size, i.e. all of these colors are used. Assume also that G is 4-colorable but not 3-colorable.

Notice that in G':

(a) All central vertices are located at distance at least 3 from each other, so they may take the same color. Thus, if this is not happening already in Φ_0, we fix Φ_0 by constructing a new assignment Φ_1 which uses at most the same number of colors as Φ_0.

(b) All existing-vertex-hanging5 vertices are at distance at least 3 from each other. Thus, we may color them by the same color. However, notice that this color should be at frequency distance at least 2 from the color in Φ_1 of the existing-vertices. We thus obtain a radiocoloring Φ_2 which is also better than Φ_1, because it does not increase the number of colors of Φ_1.

(c) Consider now all the $\Gamma + 1$ intermediate, existing-hanging, existing-hanging5 vertices in G' of an existing vertex v of degree at least 3 in G. All these vertices need to have different colors from their $\Gamma + 1$ distance two neighbors (intermediate, existing-hanging5 vertices). So, these vertices need at least $\Gamma + 1$ colors to be radiocolored. Notice, also that at least one intermediate vertex i is connected to a central vertex; thus, these colors are different from the color of the central vertices and at frequency distance 2 from that color. This implies that it leaves one color unused. But this color is used elsewhere because of our assumption of the minimality of Φ_0. So, up to now, we used $\Gamma + 3$ different colors and a span of the same size.

Note however, that v, which is a vertex with degree at least 3 in G, is a distance two neighbor in G' of at least three existing vertices (neighbors of v in G). At least one such v, unavoidably, has to demonstrate the needed 4-coloring of G.

Consider now one (let u) of those at least three existing neighbors of v in G, which takes unavoidably different colors from v and themselves, due to the 4-coloring of G. Now, look at the distance one vertices of u (intermediate, existing-hanging, existing-hanging5 vertices). Call this set $N(u)$. The corresponding set $N(v)$ can take exactly the same set of colors, else the assignment is worse. In order to radiocolor v and its 3 existing vertices, we will use four more colors. Thus, by now we need a total of $\Gamma + 5$ colors and a span of the same size.

Consider now v and one central vertex c of one of v's neighboring faces. If c gets the same color with one of the colors of $N(v)$, then it has to avoid its $\Gamma+2$ neighbors (intermediate, central-hanging, central-hanging2 vertices). Also c has to avoid v's color and two colors of v's neighbors in G; that is, it must avoid $\Gamma+5$ colors, and so it uses (including its color) $\Gamma+6$ colors and span of the same size.

If this were not so (i.e. c gets a color different from all the colors of $N(v)$) then look at v. Now, v must avoid the $\Gamma+1$ colors of $N(v)$, plus the color of c, plus (unavoidably) three colors of its existing neighbors in G. Thus, v avoids $\Gamma+5$ colors and thus, including its color, it must use $\Gamma+6$ colors and span of the same size.

In both cases we get a final assignment Φ_f (we denote it by Φ_f, where $f \geq 2$), which (may be better but) is certainly not worse than Φ_0, which needs a total of $\Gamma+6$ colors and a span of the same size. This is a contradiction to the initial assumption for Φ_0.

(End of proof of claim) □

This concludes the $\mathcal{NP}$-completeness proof for the radio coloring of a planar graph G.

(End of proof of Theorem 1) □

Corollary 4. *The min span order RCP for planar graphs is $\mathcal{NP}$-complete.* □

Very recently and independently, Bodlaender et al ([12]) proved the that $\lambda(2,1)$-labeling problem (described in the introductory Section) is $\mathcal{NP}$-complete for planar graphs. Note that this problem is equivalent to the problem of min span radiocoloring.

5.4 A Constant Ratio Approximation Algorithm

We provide here an approximation algorithm for radiocoloring of planar graphs by modifying the constructive proof of the theorem presented by Heuvel and McGuiness in [49]. Our algorithm is easier to verify with respect to correctness than what the proof given by [49] suggests. It also has better time complexity (i.e. $O(n\max\{\log n, \Delta\})$ compared to the (implicit) algorithm in [49] which is $O(n^2)$ and also assumes that a planar embedding of the graph is given. The improvement was achieved by performing the heavy part of the computation of the algorithm only in a some instances of G instead of all as in [49]. This enabled less checking and computations of the algorithm. Also, the behavior of our algorithm is very simple, straight and more time efficient for graphs of small degrees. Finally, the algorithm provided here needs no planar embedding of G, as opposed to the algorithm implied in [49].

Very recently and independently, Agnarsson and Halldórsson [1] presented approximations for the chromatic number of square and power graphs (G^k). Their method is non constructive. Thus, the algorithm implied is difficult to implement and is not efficient. Also, the performance ratio obtained is 2 for all graphs.

The theorem of [49] states that a planar graph G can be radiocolored with at most $2\Delta+25$ colors. More specifically, [49] considers the problem of L_(p,q)-Labelling, which is defined as following:

Definition 21. L_(p,q)-Labelling *Find an assignment $\phi : V \longrightarrow \{0,1,\ldots,\nu\}$, called L_(p,q)-Labelling, which satisfies: $|\phi(u)-\phi(v)| \geq p$ if $d(u,v)=1$ and $|\phi(u)-\phi(v)| \geq q$ if $d(u,v)=2$.*

Definition 22. *The minimum ν for which an L_(p,q)-Labelling exists is denoted by $\lambda(G;p,q)$ (the p,q-span of G).*

In other words, when the distance between two vertices is 1, they should be colored with colors of distance p apart, and when they are located at distance 2, they should be colored with colors of distance q apart. Note that λ_(p,q)-Labelling is a generalization of radiocoloring since λ_(p,q)-Labelling is equal to radiocoloring when p=2 and q=1. The main theorem of [49] is the following:

Theorem 12. *([49]) If G is a planar graph with maximum degree Δ and p, q are positive integers with $p \geq q$, then*

$$\lambda(G; p, q) \leq (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23.$$

By setting $p = q = 1$ and using the observation $\lambda(G; 1, 1) = \chi(G^2)$, we get immediately, as also [49] notices, that:

Corollary 5. *If G is a planar graph of maximum degree Δ, then $\chi(G^2) \leq 2\Delta + 25$.*

Theorem 12 is proved using two lemmas. We start with the first.

Lemma 10. *(Heuvel and McGuiness [49]) Let G be a simple planar graph. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $\Delta(v_1) \leq \cdots \leq \Delta(v_k)$ such that one of the following is true:*
(i) $k \leq 2$;
(ii) $k = 3$ with $\Delta(v_1) \leq 11$;
(iii) $k = 4$ with $\Delta(v_1) \leq 7$ and $\Delta(v_2) \leq 11$;
(iv) $k = 5$ with $\Delta(v_1) \leq 6$, $\Delta(v_2) \leq 7$, and $\Delta(v_3) \leq 11$.

The second Lemma is quite similar.

Lemma 11. *[49] Let G be a simple planar graph with maximum degree Δ. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $\Delta(v_1) \leq \cdots \leq \Delta(v_k)$ such that one of the following is true:*
(i) $k \leq 2$;
(ii) $k = 3$ with $\Delta(v_1) \leq 5$;
(iii) $k = 3$ with $t(vv_i) \geq 1$ for some i;
(iv) $k = 4$ with $\Delta(v_1) \leq 4$;
(v) $k = 4$ with $t(vv_i) = 2$ for some i;
(vi) $k = 5$ with $\Delta(v_i) \leq 4$ and $t(vv_i) \geq 1$ for some i;
(vii) $k = 5$ with $\Delta(v_i) \leq 5$ and $t(vv_i) = 2$ for some i;
(viii) $k = 5$ with $\Delta(v_1) \leq 7$ and $t(vv_i) \geq 1$ for all i;
(ix) $k = 5$ with $\Delta(v_1) \leq 5$, $\Delta(v_2) \leq 7$, and for each i with $t(vv_i) = 0$ it holds that $\Delta(v_i) \leq 5$
where $t(vu)$ is the number of triangular faces containing edge vu in the maximal planar graph.

These two lemmas give the so-called *unavoidable configurations* of G. The following operations apply to G:

- G/e: For an edge $e \in E$ let G/e denote the graph obtained from G by contracting e. This is the operation of collapsing an edge uv of G to a single vertex and connecting the vertex obtained to the neighbors of vertices u an v. Multiple edges are eliminated.
- $G * v$: For a vertex $v \in V$ let $G * v$ denote the graph obtained by deleting v and for each $u \in N(v)$ adding an edge between u and u^- and between u and u^+ (if these edges do not exist in G already). The notation $N(v)$ denotes the neighbors of v. The notation u^-, with $u^- \in N(v)$, denotes the edge $vu-$ which directly precedes edge vu (moving clockwise), and u^+, with $u^+ \in N(v)$, refers to the edge $vu+$ which directly succeeds edge vu.

The two lemmas and the two operations above are used to define the graph H, a vertex $v \in V(G)$ and an edge $e \in E(G)$. The main idea is to define H to be $H = G/e$ or $H = G * v$, with $e = vv_1$ and $\Delta(v) \leq 5$, depending on which cases of the two lemmas holds, so that always $\Delta(H) \leq \Delta$. Using these observations it is proved, by induction, that the minimum (p,q)-span needed for the L_(p,q)-Labelling of H is less or equal to $\lambda(G; p, q) \leq (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23$.

Assume a radiocoloring assignment of H using the above number of colors. From H, we can easily return to G as follows. If $H = G/e$ then let v' the new vertex created from the contraction of edge

$e = vv_1$. In this case, in G we set $v_1 = v'$ (this is a valid assumption since $\Delta(v_1) \leq \Delta(v')$) and set the color of v' to v_1. Now we only need to color vertex v (for both the cases of $H = G/e$ or $H = G * v$). From the way v was chosen (depending on which of the two lemmas holds), we have $\Delta(v) \leq 5$, and it is easily seen that it can be colored with one of the $\lambda(G;p,q) \leq (4q-2)\Delta + 10p + 38q - 23$ colors.

For the case of radiocoloring a planar graph G, we can use p=1 and q=1 for the order; thus, Theorem 12 states that we need at most $2\Delta + 25$ colors.

The Algorithm

We will use only Lemma 10 and the operation G/e in order to provide a much simpler and more efficient algorithm than the one implied in [49]. To see why, observe that the heavy part of the algorithm is to compute the new graph G', then to compute a radiocoloring assignment on that graph and finally to return to G. The proposed algorithm performs this part only in some instances of G instead of all as in [49]; those with maximum degree $\Delta(G) > 12$. Also, this enables less checking and computations of the algorithm. Therefore, its time complexity is substantially improved. Moreover, the behavior of the algorithm differentiates only depending on whether $\Delta(G) \leq 12$ or not. Thus, it has an almost uniform behavior and it is very simple and straight, especially, for graphs of small degrees (where it is significantly more time-efficient). Finally, the algorithm provided here needs no planar embedding of G, as opposed to the algorithm implied in [49]. A high level description of the algorithm is described here.

Algorithm Radiocoloring(G)

[I] Sort the vertices of G by their degree.
[II] If $\Delta \leq 12$ then
follow procedure (1) below:
Procedure (1): Every planar graph G has at least one vertex of degree ≤ 5.
Now, inductively assume that any proper (in vertices) subgraph of G can be radiocolored by 66 colors. Consider a vertex v in G with $\Delta(v) \leq 5$. Delete v from G to get G'. Now recursively radio color G' with 66 colors. The number of colors that v has to avoid is at most $5\Delta + 5 \leq 65$. Thus, there is one free color for v.
[III] If $\Delta > 12$ then

1. Find a vertex v and a neighbor v_1 of it, as described in Lemma 3, and set $e = vv_1$.
2. Form $G' = G/e$ ($G' = (V', E')$ with $|V'| = n - 1$, while $|V| = n$) and denote the new vertex in G' obtained by the contraction of edge e as v'.
   Modify the sorted degrees of G by deleting v, v_1, and inserting v' at the appropriate place, and also modify the possible affected degrees of the neighbors of both v and v_1.
3. $\phi(G')$ = Radiocoloring (G') (the function calls itself in order to get a radiocoloring of G').
4. Extend $\phi(G')$ to a radiocoloring of G:
   (a) Set $v_1 = v'$ and give to v_1 the color of v'.
   (b) Color v with one of the colors used in the radiocoloring ϕ of G'.

Analysis of the Algorithm

Correctness

Notice first that procedure (1) implies a coloring of G^2 with $X = 66$ colors. Then, assign the frequencies $1, 3, \ldots, 2X - 1$ to the obtained color classes of G^2. This is a radiocoloring of G with the same number of colors. To see this observe that, the only difference between the coloring of G^2 and the radiocoloring of G is the distance one constraints; in the radiocoloring any two vertices of distance one should take colors that differ by at least two instead of one as in the coloring of G^2. However, the multiplication by two of each color class of the coloring of G^2, results in a coloring of G that satisfies both distance one and two constraints of the radiocoloring. Hence, it is a radiocoloring of G.

Proposition 2. *The algorithm* Radiocoloring(G) *outputs a radiocoloring for* G *using no more than* $max\{66, 2\Delta + 25\}$ *colors.*

Proof. Assume inductively that, the recursive step 3 in [III] outputs a radiocoloring of G using at most $max\{66, 2\Delta + 25\}$ colors. Note that $\Delta(G') = \Delta(G)$, because of the way v and $e = vv_1$ are chosen.

Then, at the next Step (4) of the algorithm, the radiocoloring $\phi(G')$ of G' is extended to a radiocoloring of G, using no more colors than those used in Step 3. This extension is correct as explained here: First, (Step (a)) the vertex v_1 of G takes the color of the vertex v' of G'. This assignment is valid since v_1 has only a subset of the neighbors of v' at distance one and two.

Next, (Step (b)), the vertex v of G is colored with one of the colors used in the radiocoloring $\phi(G')$ of G'. These colors are enough for v to get a valid color. The correctness of this claim is explained below.

For any vertex $v \in V(G)$, the number of vertices at distance two from v is equal to $\sum_{u \in N(v)} \Delta(u) - \Delta(v) - 2\,t(v)$, where $t(v)$ denotes the number of triangular faces containing v in the maximal planar graph, and $\Delta(u)$ the degree of vertex u, in some planar embedding of G.

By the way v (according to Lemma 10) was chosen, it holds that $\Delta(v) \leq 5$. Also by Lemma 10, vertex v has at most two neighbors of degree at most Δ; the rest three neighbors of it have limited degree ($\leq$ 6, $\leq$ 7, $\leq$ 11). Hence, the above sum gives that there are at most $2\,\Delta + (6 + 7 + 11) - 5 = 2\,\Delta + 19$ vertices at distance two from v. Thus, the total number of distance one and distance two neighbors of v are $\leq 5 + (2\Delta + 19)$ which is

$$5 + (2\Delta + 19) < 2\Delta + 25$$

Hence, we can choose a color from the radiocoloring $\phi(G')$ of G' in order to give a valid color to v. Thus, algorithm Radiocoloring(G) gives a radiocoloring to G using no more than $max\{66, 2\Delta + 25\}$ colors. □

Time Efficiency and Approximation Ratio of the Algorithm

Lemma 12. *Algorithm* Radiocoloring(G) *approximates* $X_{order}(G)$ *by a constant factor at most* $max\{2 + \frac{25}{\Delta}, \frac{66}{\Delta}\}$.

Proof. Clearly, $X_{order}(G) > \Delta(G)$. By Proposition 1, algorithm Radiocoloring(G) uses at most $max\{66, 2\Delta + 25\}$ colors. So that,

$$1 < \frac{X_{order}(G)}{\Delta} \leq max\left\{2 + \frac{25}{\Delta}, \frac{66}{\Delta}\right\} \tag{3}$$

□

Lemma 13. *Algorithm* Radiocoloring(G) *runs in* $O(n \cdot max\{\log n, \Delta\})$ *sequential time.*

Proof. Step [I] takes $O(n \log n)$ time and Step [II] takes $O(n)$ time. Let S be the set of neighbors of both v and v_1. Each implementation of [III].1 and 2 needs time $\Delta(v) + \Delta(v_1) + \sum_{x \in S} \Delta(x)$ in order to perform the operation G/e and $O(\log n)$ time to modify the sorted degree list. The total time spent recursively is then just $O(\sum_{v \in V} \Delta(v) \cdot \log n) = O(n \cdot max\{\log n, \Delta\})$. Each implementation of [III].4 needs $O(\Delta)$ time at most and this step is executed at most n times. Thus, the total time for all executions of [III].4 is $O(n \cdot max\{\log n, \Delta\})$. This dominates the total execution time. □

A more sophisticated and efficient implementation of the contraction operation can be found in [55]. This implementation avoids the $\log n$ factor, thus making the algorithm's running time to be $O(n\Delta)$ and improves our proposed implementation when $\Delta < \log n$.

5.5 A Fully Polynomial Time Randomized Approximation Scheme (FPRAS) for the Number of Radiocolorings of a Planar Graph

Sampling and Counting

Let G be a planar graph of maximum degree $\Delta = \Delta(G)$ on vertex set $V = \{0, 1, \ldots, n-1\}$, and let C be a set of λ colors. Let $\phi : V \to C$ be a radiocoloring of the vertices of G. Such a radiocoloring always exists if $\lambda \geq 2\Delta + 25$, and it can be found by our $O(n \max\{\log n, Delta\})$ time algorithm of the Section 5.4.

Consider the Markov Chain (X_t) whose state space $\Omega = \Omega_\lambda(G)$ is the set of all radiocolorings of G with λ colors and whose transition probabilities from radiocoloring X_t are modeled as follows:

1. Choose a vertex $v \in V$ and a color $c \in C$ uniformly at random.
2. Recolor vertex v with color c. If the resulting coloring X' is a radiocoloring, then let $X_{t+1} = X'$, else $X_{t+1} = X_t$.

The procedure above it is similar to the "Glauber Dynamics" of an antiferromagnetic Potts model at zero temperature, and it was used in [50] to estimate the number of colorings of any low degree graph with k colors.

The Markov Chain (X_t), which we refer to in the sequel as $M(G, \lambda)$, is *ergodic*, provided $\lambda \geq 2\Delta + 26$, in which case its stationary distribution is *uniform* over Ω. We show here that $M(G, \lambda)$ is *rapidly mixing*, i.e., it converges, in time polynomial in n, to a close approximation of the stationary distribution, provided that $\lambda \geq 2(2\Delta + 25)$. This can be used to get a fully polynomial time randomized approximation scheme (henceforth abbreviated as FPRAS) for the number of radiocolorings of a planar graph G with λ colors in the case where $\lambda \geq 4\Delta + 50$.

Definition 23. (FPRAS) *A randomized approximation scheme for the radiocolorings, with λ colors, of a planar graph G is a probabilistic algorithm that takes as input the graph G and an error bound $\epsilon > 0$ and outputs a number Y (a random variable) such that*

$$\Pr\{(1-\epsilon)\, |\Omega_\lambda(G)| \leq Y \leq (1+\epsilon)\ |\Omega_\lambda(G)|\} \geq \frac{3}{4}$$

Such a scheme is said to be fully polynomial *if it runs in time polynomial in n and ϵ^{-1}.*

Some Definitions and Measures

For $t \in N$ let $P^t : \Omega^2 \to [0,1]$ denote the t-step transition probabilities of the Markov Chain $M(G, \lambda)$ so that $P^t(x, y) = \Pr\{X_t = y | X_0 = x\}, \forall x, y \in \Omega$. It is easy to verify that $M(G, \lambda)$ is (a) *irreducible* and (b) *aperiodic*. The irreducibility of $M(G, \lambda)$ follows from the observation that any radiocoloring x may be transformed to any other radiocoloring y by sequentially assigning new colors to the vertices V in ascending sequence; before assigning a new color c to vertex v, it is necessary to recolor all vertices $u > v$ that have color c, but there is at least one "free" color to allow this, provided $\lambda \geq 2\Delta + 26$. Aperiodicity follows from the fact that the loop probabilities are $P(x, x) \neq 0, \forall x \in R$.

Thus, the finite Markov Chain $M(G, \lambda)$ is *ergodic*, i.e., it has a stationary distribution $\pi : \Omega \to [0,1]$ such that $\lim_{t\to\infty} P^t(x, y) = \pi(y), \forall x, y \in \Omega$. Now if $\pi' : \Omega \to [0,1]$ is any function satisfying "local balance", i.e. $\pi'(x)P(x, y) = \pi'(y)P(y, x)$ then if $\sum_{x\in\Omega} \pi'(x) = 1$, π' is indeed the stationary distribution. In our case, $P(y, x) = P(x, y)$; thus the stationary distribution of $M(G, \lambda)$ is *uniform*.

The efficiency of any approach like this to sample radiocolorings crucially depends on the rate of convergence of $M(G, \lambda)$ to stationarity. There are various ways to define closeness to stationarity, but all are essentially equivalent in this case, and we will use the "variation distance" at time t with respect to initial vertex x

$$\delta_x(t) = \max_{S \subseteq \Omega} \left|P^t(x, S) - \pi(S)\right| = \frac{1}{2} \sum_{y \in \Omega} \left|P^t(x, y) - \pi(y)\right|$$

where $P^t(x,S) = \sum_{y \in S} P^t(x,y)$ and $\pi(S) = \sum_{x \in S} \pi(x)$.

Note that this is a *uniform bound* over all events $S \subseteq R$ of the difference of probabilities of event S under the stationary and t-step distributions. The *rate of convergence to stationarity* starting from initial vertex x is

$$\tau_x(\epsilon) = \min\{t : \delta_x(t') \leq \epsilon, \forall t' \geq t\}$$

This informally means that at time $t = \tau_x(\epsilon)$ the Markov Chain's t-steps distribution is "at distance at most ϵ" from the stationary distribution.

Rapid Mixing

As indicated by the (by now standard) techniques for showing rapid mixing by *coupling* ([50, 51]), our strategy here is to construct a coupling for $M = M(G, \lambda)$, i.e., a stochastic process (X_t, Y_t) on $\Omega \times \Omega$ such that each of the processes (X_t), (Y_t), considered in isolation, is a faithful copy of M. We will arrange a joint probability space for (X_t), (Y_t) so that, apart from being independent, the two processes tend to *couple* so that $X_t = Y_t$ for t large enough. If coupling can occur rapidly (independently of the initial states X_0, Y_0), we can infer that M is rapidly mixing, because the variation distance of M from the stationary distribution is bounded above by the probability that (X_t) and (Y_t) *have not* coupled by time t (see e.g., [3, 24]).

The transition $(X_t, Y_t) \rightarrow (X_{t+1}, Y_{t+1})$ in the coupling is defined by the following experiment:

1. Select $v \in V$ uniformly at random (u.a.r.).
2. Compute a permutation $g(G, X_t, Y_t)$ of C according to the following procedure: The procedure to compute $g(G, X_t, Y_t)$ is as follows:
   (a) If $v \in D$ then g is the identity.
   (b) If $v \in A$ then proceed as follows: Denote by N the set of neighbors of v in G^2. Define $C_x \subseteq C$ to be the set of all colors c, such that some vertex in N receives c in radiocoloring Y_t but no vertex in N receives c in radiocoloring Y_t. Let C_y be defined as C_x with the roles of X_t, Y_t interchanged. Observe $C_x \cap C_y = \emptyset$ and $|C_x|, |C_y| \leq d'(v)$. Let, without loss of generality, $|C_x| \leq |C_y|$. Choose any subset $C'_y \subseteq C_y$ with $|C'_y| \leq |C_x|$ and let $C_x = \{c_1, \ldots, c_r\}, C'_y = \{c'_1, \ldots, c'_r\}$ be enumerations of $C_x, C_{y'}$ resulting from the orderings of X_t, Y_t. Let g be the permutation $(c_1, c'_1), \ldots, (c_r, c'_r)$ which interchanges the color sets $C_x, C_{y'}$ and leaves all other colors fixed.
3. Choose a color $c \in C$ u.a.r.
4. In the radiocoloring X_t (resp., Y_t) recolor vertex v with color c (resp., $g(c)$) to get a new radiocoloring X' (resp., Y')
5. If X' (resp., Y') is a radiocoloring then $X_{t+1} = X'$ (resp., $Y_{t+1} = Y'$), else let $X_{t+1} = X_t$ (resp., $Y_{t+1} = Y_t$).

Note that, whatever procedure is used to select the permutation g, the distribution of $g(c)$ is *uniform* (because of the random uniform selection of c), thus (X_t) and (Y_t) are both faithful copies of M.

We remark that any set of vertices $F \subseteq V$ can have the same color in the graph G^2 only if they can have the same color in some radiocoloring of G. Thus, given a coloring of G^2 with λ' colors, we can construct a radiocoloring of G by giving the values (new colors) $1, 3, \ldots, 2\lambda' - 1$ in the color classes of G^2. Note that this transformation preserves the number of colors (but not the span).

Now let $A = A_t \subseteq V$ be the set of vertices on which the colorings of G^2 implied by X_t, Y_t agree and $D = D_t \subseteq V$ be the set on which they disagree. Let $d'(v)$ be the number of edges incident at v in G^2 that have one point in A and one in D. Clearly, if m' is the number of edges of G^2 spanning A and D, we get $\sum_{v \in A} d'(v) = \sum_{v \in D} d'(v) = m'$.

It is clear (since in each transition at most one vertex is recolored) that $|D_{t+1}| - |D_t| \in \{-1, 0, 1\}$.

(i) Consider first the probability that $|D_{t+1}| = |D_t| + 1$. For this event to occur, the vertex v selected in step (1) of the procedure for g must lie in A, and hence we go by step 2.(b) of the experiment described above. If the new radiocolorings are to disagree at vertex v then the color c selected in line (3) must be an element of C_y. But $|C_y| \leq d'(v)$; hence

$$\Pr\{|D_{t+1}| = |D_t| + 1\} \leq \frac{1}{n} \sum_{v \in A} \frac{d'(v)}{\lambda} = \frac{m'}{\lambda_n} \tag{4}$$

(ii) Now consider the probability that $|D_{t+1}| = |D_t| - 1$. For this to occur, the vertex v must lie in D; hence, the permutation g selected in line (2) is the identity. For X_{t+1} and Y_{t+1} to agree at v, it is enough that color c selected in step (3) is different from all the colors that X_t and Y_t imply for the neighbors of v in G^2. The number of colors c that satisfy this, is (by our previous arguments) at least $\lambda - 2(2\Delta + 25) + d'(v)$; hence,

$$\begin{aligned} \Pr\{|D_{t+1}| = |D_t| - 1\} &\geq \frac{1}{n} \sum_{v \in D} \frac{\lambda - 2(2\Delta + 25) + d'(v)}{\lambda} \\ &\geq \frac{\lambda - 2(2\Delta + 25)}{\lambda n} \, |D| + \frac{m'}{\lambda n} \end{aligned} \tag{5}$$

Define now $\alpha = \frac{\lambda - 2(2\Delta+25)}{\lambda n}$ and $\beta = \frac{m'}{\lambda n}$. Then

$$\Pr\{|D_{t+1}| = |D_t| + 1\} \leq \beta$$

and $\Pr\{|D_{t+1}| \geq |D_t| - 1\} \geq \alpha |D_t| + \beta$

Given $\alpha > 0$, i.e., $\lambda > 2(2\Delta + 25)$, Equations (4) and (5) imply that

$$\begin{aligned} E(|D_{t+1}|) &\leq \beta(|D_t| + 1) + (\alpha|D_t| + \beta)(|D_t| - 1) + (1 - \alpha|D_t| - 2\beta)|D_t| \\ &\leq (1 - \alpha)|D_t| \end{aligned} \tag{6}$$

given $|D_t|$.

Thus, from Bayes, we get $E(|D_{t+1}| \text{ given } |D_0|) \leq (1-\alpha)^t |D_0| \leq n(1-\alpha)^t$

and since $|D_t|$ is a non-negative random variable, we get, by the Markov inequality, that

$$\Pr\{D_t \neq 0\} \leq n(1-\alpha)^t r \leq n e^{-\alpha t}$$

since, for any non-negative integer random variable X the Markov inequality states that

$$\Pr[X > 0] \leq E(X)$$

So, we note that $\Pr\{D_t \neq \emptyset\} \leq \epsilon$ provided that $t \geq \frac{1}{\alpha} \ln\left(\frac{n}{\epsilon}\right)$, thus proving:

Theorem 13. *Let G be a planar graph of maximum degree Δ on n vertices. Assuming $\lambda \geq 2(2\Delta+25)$ the convergence time $\tau(\epsilon)$ of the Markov Chain $M(G, \lambda)$ is bounded above by,*

$$\tau_x(\epsilon) \leq \frac{\lambda}{\lambda - 2(2\Delta + 25)} \; n \ln\left(\frac{n}{\epsilon}\right)$$

regardless of the initial state x.

□

An FPRAS for Radiocolorings with λ Colors

The technique we employ is as in [50], which is fairly standard in the area. By using it, we obtain:

Theorem 14. *There is a fully-polynomial, randomized approximation scheme for the number of radiocolorings of a planar graph G with λ colors, provided that $\lambda > 2(2\Delta+25)$, where Δ is the maximum degree of G.*

Proof. Recall that $\Omega_\lambda(G)$ is the set of all radiocolorings of G, with λ colors. Let m be the number of edges in G, and let

$$G = G_m \supseteq G_{m-1} \supseteq \cdots \supseteq G_1 \supseteq G_0$$

be any sequence of graphs where G_{i-1} is obtained by G_i by removing a single edge. We can always erase an edge whose one node is of degree at most 5 in G_i. Clearly,

$$|\Omega_\lambda(G)| = \frac{|\Omega_\lambda(G_m)|}{|\Omega_\lambda(G_{m-1})|} \cdot \frac{|\Omega_\lambda(G_{m-1})|}{|\Omega_\lambda(G_{m-2})|} \cdots \frac{|\Omega_\lambda(G)|}{|\Omega_\lambda(G_0)|} \cdot |\Omega_\lambda(G_0)|$$

But $|\Omega_\lambda(G_0)| = \lambda^n$ for all kinds of colorings. The standard strategy is to estimate the ratio

$$\rho_i = \frac{|\Omega_\lambda(G_i)|}{|\Omega_\lambda(G_{i-1})|}$$

for each i, $1 \leq i \leq m$.

Suppose that graphs G_i, G_{i-1} differ in the edge uv which is present in G_i but not in G_{i-1}. Clearly, $\Omega_\lambda(G_i) \subseteq \Omega_\lambda(G_{i-1})$. Any radiocoloring in $\Omega_\lambda(G_{i-1}) \setminus \Omega_\lambda(G_i)$ assigns either the same color to u and v or the color values of u and v differ by only 1. Let $\Delta(v) \leq 5$ in G_i. So, we now have to recolor u with one of at least $\lambda - (2\Delta + 35) = 2\Delta + 15$, colors (from section 5.4 of this paper). Each radiocoloring of $\Omega_\lambda(G_i)$ can be obtained in at most one way by algorithm Radiocoloring(G) of the Section 5.4 as the result of such a perturbation; thus,

$$\frac{2\Delta + 15}{2(\Delta + 1) + 15} \leq \rho_i < 1 \tag{7}$$

To avoid trivialities, assume $0 < \epsilon \leq 1, n \geq 3$ and $\Delta > 2$. Let $Z_i \in \{0, 1\}$ be the random variable obtained by simulating the Markov Chain $M(G_{i-1}, \lambda)$ from any certain fixed initial state for

$$T = \frac{\lambda}{\lambda - 2(2\Delta + 25)} \, n \ln \left(\frac{4nm}{\epsilon} \right)$$

steps, and returning to 1 if the final state is a member of $\Omega_\lambda(G_i)$ and 0 else. Let $\mu_i = E(Z_i)$. By Theorem 13, we have

$$\rho_i - \frac{\epsilon}{4m} \leq \mu_i \leq \rho_i + \frac{\epsilon}{4m}$$

and by Equation (7), we get

$$\left(1 - \frac{\epsilon}{2m}\right) \rho_i \leq \mu_i \leq \left(1 + \frac{\epsilon}{2m}\right) \rho_i$$

As our estimator for $|\Omega_\lambda(G)|$, we use

$$Y = \lambda^n Z_1 Z_2 \cdots Z_m$$

Note that $E(Y) = \lambda^n \mu_1 \mu_2 \cdots \mu_m$. But

$$Var(Y) \leq \frac{Var(Z_1 Z_2 \cdots Z_m)}{(\mu_1 \mu_2 \cdots \mu_m)^2} = \prod_{i=1}^{m} \left(1 + \frac{Var(Z_i)}{\mu_i^2}\right) - 1$$

By standard techniques (as in [50]), one can easily show that Y satisfies the requirements for an FPRAS for the number of radiocolorings with λ colors $|\Omega_\lambda(G)|$. □

6 Further work

A major open problem is to obtain tighter bounds for the chromatic number of the square of a planar graph. Also, to get a polynomial time approximation to $X_{order}(G)$ of asymptotic ratio < 2. The improvement of the time efficiency of the approximation procedure is also a subject of further work.

Also, another interesting open problem is to get better approximations for the min span radiocoloring problem, especially for planar graphs.

Furthermore, it is interesting to investigate whether one can reduce λ and still get an FPRAS for the number of radiocolorings of a graph with λ colors.

With respect to the Radio Labelling problem, an interesting direction for further research is to prove (or disprove) the conjecture that, given a graph $G(V, E)$ and a partition of V into κ cliques, G is Hamiltonian iff there exists a Hamiltonian Cycle containing at most $2(\kappa - 1)$ inter-clique edges. This would imply a $n^{\mathcal{O}(\kappa)}$ algorithm for the Radio Labelling of a graph $\overline{G}$, given a coloring of $\overline{G}$ with κ colors.

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