The Price of Defense*

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Abstract

We consider a *strategic game* on a graph $G(V, E)$ with two confronting classes of randomized players: ν *attackers* who choose vertices and wish to minimize the probability of being caught by the *defender*, who chooses edges and gains the expected number of attackers it catches. So, the defender captures system rationality. In a *Nash equilibrium*, no single player has an incentive to unilaterally deviate from its randomized strategy. The *Price of Defense* is the worst-case ratio, over all Nash equilibria, of the *optimal* gain of the defender (which is ν) over the gain of the defender at a Nash equilibrium. In this work, we provide a comprehensive collection of trade-offs between the Price of Defense and the computational efficiency of Nash equilibria.

- Through a reduction to a *Zero-Sum Two-Players Game*, we prove that a general Nash equilibrium can be computed via Linear Programming in polynomial time. However, the reduction does not provide any apparent guarantees on the Price of Defense.
- To obtain guarantees on Price of Defense, we analyze several structured Nash equilibria:
  - In a *Matching Nash Equilibrium*, the support of the defender is an *Edge Cover* of the graph. We prove that Matching Nash equilibria can still be computed in polynomial time, and they incur a Price of Defense of $\alpha(G)$, the *Independence Number* of G.
  - In a *Perfect Matching Nash Equilibrium*, the support of the defender is a *Perfect Matching* of the graph. We prove that Perfect Matching Nash Equilibria can be computed in polynomial time, and they incur a Price of Defense of $\frac{|V|}{2}$.
  - In a *Defender Uniform Nash equilibrium*, the defender chooses each edge in its support with uniform probability. We prove that Defender Uniform Nash equilibria incur a Price of Defense falling between those for Matching and Perfect Matching Nash Equilibria; however, it is $\mathcal{NP}$-complete to even decide the existence of a Defender Uniform Nash equilibrium.
  - In an *Attacker Symmetric, Uniform Nash equilibrium*, all attackers have a common support on which each uses a uniform probability distribution. We prove that Attacker Symmetric Uniform Nash equilibria can be computed in polynomial time and incur a Price of Defense of either $\frac{|V|}{2}$ or $\alpha(G)$.

In conclusion, the Perfect Matching Nash Equilibrium both can be computed efficiently and provides the best (known) Price of Defense, when it exists. Else, Matching Nash equilibria, Defender Uniform Nash equilibria and Attacker Symmetric Uniform Nash equilibria provide interesting trade-offs between the Price of Defense and computational efficiency.

Throughout the paper, missing proofs can be found in the attached Appendix. It may be read at the discretion of the Program Committee.

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1 Introduction

Motivation and Framework. We revisit a network game with *attackers* and a *defender*, introduced recently by Mavronicolas *et al.* [12] and further studied in [7, 13]; the game was conceived as an appropriate theoretical model of security *attacks* and *defenses* in emerging networks like the Internet. In this network game, nodes are vulnerable to infection by *threats*, called *attackers*. Available to the network is a security software (or *firewall* [3]), called the *defender*, cleaning some part of the network.

This network game is partially motivated by *Network Edge Security* [11], a new distributed firewall architecture where a firewall is implemented in a *distributed* way and protects the subnetwork spanned by the nodes participating in the distributed implementation. The simplest case where the subnetwork is a single link (with its two incident nodes) offers the initial basis for the theoretical model of Mavronicolas *et al.* [12]. Understanding the mathematical pitfalls of attacks and defenses in this simplest theoretical model is a necessary prerequisite for making rigorous progress in the analysis of distributed firewall architectures with more involved topologies.

Each attacker (called a *vertex player*) targets a *node* of the network chosen via its own probability distribution on nodes; the defender (called the *edge player*) chooses a single *link* via its own probability distribution on links. A node chosen by an attacker is destroyed unless it crosses the link being cleaned by the defender. The *Individual Profit* of an attacker is the probability that it is not caught; the *Individual Profit* of the defender is the expected number of attackers it catches.

To the best of our knowledge, the network game of Mavronicolas *et al.* [12] is the *first* strategic game where the network (*system*) is *explicitly* modeled as a distinct, non-cooperative player (namely, the defender). Unlike previously studied games that evaluated the effect of selfish behavior on system performance using the *Price of Anarchy* [10, 18] (which *implicitly* modeled the system), we pursue this evaluation by defining and using the *Price of Defense* as the worst case ratio of ν over the Individual Profit of the defender.

We are interested in analyzing the Price of Defense for *Nash equilibria* [14, 15], where no single player has an incentive to deviate from its randomized strategy. (It is known [12] that pure Nash equilibria do *not* exist for this game unless the graph is trivial.) How does the Price of Defense vary with Nash equilibria? Are there Nash equilibria that both are computationally tractable and offer good Price of Defense? Such questions are the focus of our work. Our answers make a comprehensive collection of trade-offs between Price of Defense and the computational complexity of Nash equilibria.

Contribution. We prove that a (mixed) Nash equilibrium for our network game can be computed in polynomial time (Theorem 4.1). The proof is by reduction to the case of two players (one attacker and one defender), which is shown to be *constant-sum*. *Constant-sum Two-Players* games are reducible to *Linear Programming* [16], hence solvable in polynomial time [9]. However, the reduction to Linear Programming hides the Price of Defense. This invites the consideration of special classes of Nash equilibria with sufficient structure for the evaluation of the incurred Prices of Defense.

Matching Nash Equilibria. Introduced in [12], a *Matching Nash equilibrium* is defined to satisfy several (necessary) *covering* properties of Nash equilibria (for example, the support of the edge player is an *Edge Cover* of the graph) and two additional properties (for example, the support of the vertex players is an *Independent Set* of the graph), in addition, all vertex players use a common probability distribution, and each player uses a uniform probability distribution on its support.

- We provide a new characterization of graphs admitting Matching Nash Equilibria (Theorem 5.3). Such graphs have their *Independence Number* equal to their *Edge Covering Number*. The characterization improves an earlier one (stated here as Theorem 2.2) from [12]. The characterization benefits from an improved understanding of structural (graph-theoretic) properties of Matching Nash equilibria. In particular, we prove that in a Matching Nash equilibrium, the support of the vertex players is a *Maximum Independent Set* of the graph (Proposition 5.1) and the support of the edge player is a *Minimum Edge Cover* of the graph (Proposition 5.2).

- We translate the characterization into a polynomial time algorithm to (decide the existence of and) compute a Matching Nash equilibrium (Theorem 5.4). This relies on obtaining a polynomial time algorithm for the (new) graph-theoretic problem of deciding, given a graph G, whether its *Independence Number* $\alpha(G)$ and *Edge Covering Number* $\beta'(G)$ are equal, and yielding, if so, a Maximum Independent Set for the graph (Proposition 3.1). In turn, the graph-theoretic algorithm relies on computing a *Minimum Edge Cover* (via computing a *Maximum Matching*) and a subsequent reduction to $\mathsf{2SAT}$.

- We prove that the Price of Defense for a Matching Nash equilibrium is $\alpha(G)$ (Proposition 5.5). This relies on its modeling assumption that all vertex players are symmetric and uniform, and on its shown property that the support of the vertex players is a Maximum Independent Set.

Perfect Matching Nash Equilibria. A *Perfect Matching Nash equilibrium* is a Matching Nash equilibrium where, additionally, the support of the edge player is a *Perfect Matching* of the graph.

- We provide a characterization of graphs admitting a Perfect Matching Nash Equilibria (Theorem 6.2). Such graphs have a Perfect Matching and their Independence Number equals $\frac{|V|}{2}$ (V is the vertex set). The characterization benefits from a structural (graph-theoretic) property of Perfect Matching Nash equilibria we prove, namely that the support of the vertex players has size $\frac{|V|}{2}$ (Proposition 6.1).

- We translate the characterization into a polynomial time algorithm to (decide the existence and) compute a Perfect Matching Nash equilibrium (Theorem 6.3). This relies on obtaining a polynomial time algorithm for the (new) graph-theoretic problem of deciding, given a graph G with a Perfect Matching, whether its Independence Number equals $\frac{|V|}{2}$, and yielding, if so, a Maximum Independent Set for the graph (Proposition 3.2). In turn, the graph-theoretic algorithm relies on computing a Perfect Matching and a subsequent reduction to $\mathsf{2SAT}$.

- We prove that the Price of Defense for a Perfect Matching Nash Equilibrium is $\frac{|V|}{2}$ (Theorem 6.4). This relies on its modeling assumption that all vertex players are symmetric and uniform, and on its shown property that the support of the vertex players has size $\frac{|V|}{2}$.

The relation between the Prices of Defense for Perfect Matching and Matching Nash Equilibria is precisely the relation between $\frac{|V|}{2}$ and $\alpha(G)$ for the graph G. For graphs that have both Matching and Perfect Matching Nash Equilibria, Theorem 6.2 implies that $\alpha(G) = \frac{|V|}{2}$ and the two Prices of Defense coincide (as also do the two classes of equilibria). Consider a graph that has a Matching Nash Equilibrium but not a Perfect Matching Nash Equilibrium. By the characterization of Matching Nash Equilibria in [12, Theorem 3] (or Theorem 2.2 here), $\alpha(G) \geq \frac{|V|}{2}$ (else, there could not be enough vertices inside an Independent Set to which vertices outside have to be matched). Thus, the Price of Defense of a Perfect Matching Nash Equilibrium may not exceed that of a Matching Nash Equilibrium.

Defender Uniform Nash Equilibria. In a *Defender Uniform Nash Equilibrium*, the defender chooses each edge in its support with uniform probability. Such equilibria are inspired by the recent *Uniform Nash equilibria* introduced by Bonifaci *et. al.* [1, 2] for (classes of) bimatrix games. Bonifaci *et al.* [1, 2] proved that deciding the existence of Uniform Nash equilibria is $\mathcal{NP}$-complete. Clearly, our Matching and Perfect Matching Nash Equilibria are themselves Defender Uniform.

- We provide a characterization of graphs admitting Defender Uniform Nash equilibria (Theorem 7.1). The characterization involves *Regular Subgraphs*, Independent Sets and *Expanders.* (Remarkably, Regular Subgraphs were also encountered in the work of Bonifaci *et al.* [1, 2].)

Equilibrium class	Complexity	Price of Defense
Perfect Matching	$O\left(\sqrt{\lvert V\rvert}\lvert E\rvert \cdot \log_{\lvert V\rvert} \frac{\lvert V\rvert^2}{\lvert E\rvert}\right)$	$\frac{\lvert V\rvert}{2}$
Matching	$O\left(\sqrt{\lvert V\rvert}\lvert E\rvert \cdot \log_{\lvert V\rvert} \frac{\lvert V\rvert^2}{\lvert E\rvert}\right)$	$\alpha(G)$
Defender Uniform	$\mathcal{NP}$-complete	$\lvert V_i'\rvert + \frac{\lvert V_r'\rvert}{2}$ where $\lvert V_i'\rvert$, $\lvert V_r'\rvert$ are as in Theorem 7.1
Attacker Symmetric Uniform	Polynomial	$\frac{\lvert V\rvert}{2}$ or $\alpha(G)$
General	Polynomial	?

Table 1: Summary of results.

- We prove that deciding the existence of a Defender Uniform Nash equilibrium is $\mathcal{NP}$-complete (Theorem 7.2). This employs a reduction from the PARTITION INTO HAMILTONIAN SUBGRAPHS problem [6, Problem GT13], originally shown $\mathcal{NP}$-complete by Valiant [19]. A significant milestone of our reduction is a reduction to a certain undirected version of the problem. This version is also cited in [6, GT13] as $N\mathcal{P}$-complete with attribution to (personal communication with) Papadimitriou [17]. Our $\mathcal{NP}$-completeness result strengthens the corresponding $\mathcal{NP}$-completeness results of Bonifaci *et al.* [1, 2] since it applies to a *specific* game.

- We prove that the Price of Defense for Defender Uniform Nash equilibrium is $\lvert V_i'\rvert + \frac{\lvert V_r'\rvert}{2}$, where V_i' and V_r' are suitable vertex sets determined by the characterization (Theorem 7.1). We argue that this value is always between those for Perfect Matching and Matching Nash equilibria.

Compared with Matching Nash Equilibria, Defender Uniform Nash Equilibria provide a sometimes better Price of Defense, but they are hard to compute (unless $\mathcal{P} = \mathcal{NP}$). This represents an interesting trade-off for these two structured classes of Nash equilibria.

Attacker Symmetric Uniform Nash Equilibria. In an *Attacker Symmetric Uniform Nash Equilibrium*, there is a common support for attackers and each attacker uses a uniform probability distribution on it. This class is also inspired by the Uniform Nash equilibria of Bonifaci *et. al.* [1, 2].

- We provide a characterization of graphs admitting Attacker Symmetric Uniform Nash equilibria (Theorem 7.1). Roughly speaking, such graphs either allow the definition of a certain probability distribution on their edge set (Condition (1)), or have their Independence Number equal to their Edge Covering Number (Condition (2)) and hence, by Theorem 5.3, admit Matching Nash equilibria. Thus, the characterization partitions the class of Attacker Symmetric Uniform Nash equilibria into two subclasses.

- We translate the characterization into a polynomial time algorithm to (decide the existence and) compute an Attacker Symmetric Uniform Nash equilibrium (Theorem 8.2). This is in contrast to the $\mathcal{NP}$-completeness for the relative class of Defender Uniform Nash equilibria.

- We prove that the Price of Defense for Attacker Symmetric Uniform Nash Equilibria is either $\frac{\lvert V\rvert}{2}$ or $\alpha(G)$ (Theorem 8.3). The two values are implied by Conditions (1) and (2), respectively, in the characterization.

Our work reveals a case of strong interplay between Game Theory and Graph Theory. In fact, in this case, the structure of the graph has been discovered to be shaping the Nash equilibrium of choice. This shaping suggests certain ways of network design (for example, provide the availability of Perfect Matchings) in order to induce Nash equilibria with sufficiently small Price of Defense. Our results are summarized in Table 1.

2 Background, Definitions and Preliminaries

Graph Theory. Throughout, we consider an undirected graph $G = G(V, E)$ with no isolated vertices. For a vertex set $U \subseteq V$, denote $G(U)$ the subgraph of G induced by U. For an edge set $F \subseteq E$, denote $G(F)$ the subgraph of G induced by F. For a vertex $v \in V$, denote $d_G(v)$ the degree of vertex v in G, and Δ_G and δ_G the maximum and minimum, respectively, degree of the graph.

- A vertex set $IS \subseteq V$ is an *Independent Set* of the graph G if for all pairs of vertices $u, v \in IS$, $(u, v) \notin E$. A *Maximum Independent Set* of G is one that has maximum size; denote $\alpha(G)$ the size of a Maximum Independent Set of G. $\alpha(G)$ is called the *Independence Number*.
- A *Vertex Cover* of G is a vertex set $VC \subseteq V$ such that for each edge $(u, v) \in E$ either $u \in VC$ or $v \in VC$. A *Minimum Vertex Cover* of G is one that has minimum size; denote $\beta(G)$ the size of a Minimum Vertex Cover of G. $\beta(G)$ is called the *Vertex Cover Number*.
- An *Edge Cover* of G is an edge set $EC \subseteq E$ such that for every vertex $v \in V$, there is an edge $(v, u) \in EC$. A *Minimum Edge Cover* of G is one that has maximum size; denote $\beta'(G)$ the size of a Minimum Edge Cover of G. It is known that a Minimum Edge Cover consists of vertex-disjoint star graphs. (In a *star graph*, a distinguished vertex called *center* is connected to all other vertices, called *terminals*). $\beta'(G)$ is called the *Edge Covering Number*.
- A *Matching* of G is a set $M \subseteq E$ of non-incident edges. A *Maximum Matching* of G is one that has maximum size; $\alpha'(G)$ denotes the size of a Maximum Matching of G and it is called the *Matching Number*. The currently fastest algorithm to compute a Maximum Matching of G appears in [8] and has running time $O\left(\sqrt{|V|}|E| \cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$. It is known that a Minimum Edge Cover can be computed in polynomial time via computing a Maximum Matching. (See, e.g., [20, page 115].) A *Perfect Matching* is a Matching that is also an Edge Cover.

For a graph G, it trivially holds that $\alpha(G) + \beta(G) = |V|$, while also $\alpha'(G) + \beta'(G) = |V|$ (*Gallai's Theorem* [5]). Also, $\alpha'(G) \leq \beta(G)$, since a Vertex Cover of G must include at least one vertex incident to each edge in a Matching of G. It follows that $\alpha(G) \leq \beta'(G)$.

Fix now a vertex set $U \subseteq V$. The graph G is a *U-Expander* graph (and the set U is an *Expander* for G) if for each set $U' \subseteq U$, $|U'| \leq |\mathsf{Neigh}_G(U') \cap (V \backslash U)|$. An *Expanding Independent Set* [12] of the graph G is an Independent Set IS of G such that $V \backslash IS$ is an Expander for G.

Game Theory. Associated with G is a *strategic game* $\Pi(G) = \langle \mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{\mathsf{IP}\}_{i \in \mathcal{N}} \rangle$ on G:

- The set of *players* is $\mathcal{N} = \mathcal{N}_{vp} \cup \mathcal{N}_{ep}$, where $\mathcal{N}_{vp}$ has ν *vertex* players vp_i, called *attackers*, $1 \leq i \leq \nu$ and $\mathcal{N}_{ep}$ has *edge* player ep, called *defender*.
- The *strategy set* S_i of vertex player vp_i is V, and the *strategy set* S_{ep} of the edge player ep is E. So, the *strategy set* $\mathcal{S}$ of the game is $\mathcal{S} = \left(\times_{i \in N_{vp}} S_i \right) \times S_{ep} = V^{\nu} \times E$.
- Fix any *strategy profile* $\mathbf{s} = \langle s_1, \ldots, s_\nu, s_{ep} \rangle \in \mathcal{S}$, also called a *pure profile*.
  - The *Individual Profit* of vertex player vp_i is a function $\mathsf{IP}_{\mathbf{s}}(i) : \mathcal{S} \rightarrow \{0, 1\}$ such that $\mathsf{IP}_{\mathbf{s}}(i) = \begin{cases} 0, & s_i \in s_{ep} \\ 1, & s_i \notin s_{ep} \end{cases}$; intuitively, the vertex player vp_i receives 1 if it is not caught by the edge player, and 0 otherwise.
  - The *Individual Profit* of the edge player ep is a function $\mathsf{IP}_{\mathbf{s}}(ep) : \mathcal{S} \rightarrow \mathbb{N}$ such that $\mathsf{IP}_{\mathbf{s}}(ep) = |\{i : s_i \in s_{ep}\}|$; intuitively, the edge player ep receives the number of vertex players it catches.

A *mixed strategy* for player $i \in \mathcal{N}$ is a probability distribution over S_i; thus, a mixed strategy for a vertex player (resp., edge player) is a probability distribution over vertices (resp., edges) of G. A *profile* $\mathbf{s} = \langle s_1, \ldots, s_\nu, s_{ep} \rangle$ is a collection of mixed strategies; $s_i(v)$ is the probability that vertex player vp_i chooses vertex v and $s_{ep}(e)$ is the probability that the edge player ep chooses edge e.

The *support* of player $i \in \mathcal{N}$ in the mixed profile $\mathbf{s}$, denoted $\mathsf{Support}_{\mathbf{s}}(i)$, is the set of pure strategies in its strategy set to which i assigns a strictly positive probability in $\mathbf{s}$. Denote $\mathsf{Support}_{\mathbf{s}}(vp) = \bigcup_{i \in \mathcal{N}_{vp}} \mathsf{Support}_{\mathbf{s}}(i)$. Set $\mathsf{Edges}_{\mathbf{s}}(v) = \{(u,v) \in E : (u,v) \in \mathsf{Support}_{\mathbf{s}}(ep)\}$. So, $\mathsf{Edges}_{\mathbf{s}}(v)$ contains all edges incident to v that are included in the support of the edge player.

A profile is *uniform* if each player uses a uniform probability distribution on its support. For a uniform profile $\mathbf{s}$, for each vertex player $vp_i \in \mathcal{N}_{vp}$, for each vertex $v \in V$, $s_i(v) = \frac{1}{|\mathsf{Support}_{\mathbf{s}}(i)|}$; for the edge player ep, for each $e \in E$, $s_{ep}(e) = \frac{1}{|\mathsf{Support}_{\mathbf{s}}(ep)|}$. A profile $\mathbf{s}$ is *attacker symmetric* if for all vertex players vp_i, $vp_k \in \mathcal{N}_{vp}$, $\mathsf{Support}_s(i) = \mathsf{Support}_s(k)$. An *attacker symmetric uniform* profile is an attacker symmetric profile where each attacker uses a uniform probability distribution. A profile is *defender uniform* if the edge player uses a uniform probability distribution on its support.

For a vertex $v \in V$, the probability the edge player ep chooses an edge that contains the vertex v is denoted $P_{\mathbf{s}}(\mathsf{Hit}(v))$. Clearly, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} s_{ep}(e)$. For a vertex $v \in V$, denote as $\mathsf{VP}_{\mathbf{s}}(v)$ the expected number of vertex players choosing vertex v according to $\mathbf{s}$; so, $\mathsf{VP}_{\mathbf{s}}(v) = \sum_{i \in \mathcal{N}_{vp}} s_i(v)$. Also, in an attacker symmetric uniform profile $\mathbf{s}$, for a vertex $v \in \mathsf{Support}_{\mathbf{s}}(vp)$, $\mathsf{VP}_{\mathbf{s}}(v) = \sum_{i \in \mathcal{N}_{vp}} s_i(v) = \frac{\nu}{|\mathsf{Support}_s(vp)|}$. For each edge $e = (u,v) \in E$, $\mathsf{VP}_{\mathbf{s}}(e)$ is the expected number of vertex players choosing either the vertex u or the vertex v.

A mixed profile $\mathbf{s}$ induces an *Expected Individual Profit* $\mathsf{IP}_{\mathbf{s}}(i)$ for each player $i \in \mathcal{N}$, which is the expectation according to $\mathbf{s}$ of the Individual Profit of player i. For the vertex player $vp_i \in \mathcal{N}_{vp}$, $\mathsf{IP}_{\mathbf{s}}(i) = \sum_{v \in V} s_i(v) \cdot \left(1 - \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} s_{ep}(e)\right)$; for the edge player ep, $\mathsf{IP}_{\mathbf{s}}(ep) = \sum_{e=(u,v) \in E} s_{ep}(e) \cdot \left(\sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v))\right)$.

The mixed profile $\mathbf{s}$ is a *(mixed) Nash equilibrium* [14, 15] (abbreviated as *NE*) if for each player $i \in \mathcal{N}$, it maximizes $\mathsf{IP}_{\mathbf{s}}(i)$ over all mixed profiles $\mathbf{t}$ that differ from $\mathbf{s}$ only with respect to the mixed strategy of player i. By Nash's result [14, 15], there is at least one NE. In a NE, for each vertex player vp_i, for any vertex $v \in \mathsf{Support}_{\mathbf{s}}(i)$, $\mathsf{IP}_{\mathbf{s}}(i) = 1 - \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} s_{ep}(e)$; for the edge player ep, for any edge $(u,v) \in \mathsf{Support}_{\mathbf{s}}(ep)$, $\mathsf{IP}_{\mathbf{s}}(ep) = \sum_{i \in \mathcal{N}_{vp}} (s_i(u) + s_i(v))$. We use a characterization of NE from [12]:

Theorem 2.1 ([12]) *A profile* $\mathbf{s}$ *is a Nash equilibrium if and only if* (1) *for each vertex* $v \in \mathsf{Support}_{\mathbf{s}}(vp)$, $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \min_{v' \in V} P_{\mathbf{s}}(\mathsf{Hit}(v'))$, & (2) *for each edge* $e \in \mathsf{Support}_{\mathbf{s}}(ep)$, $\mathsf{VP}_{\mathbf{s}}(e) = \max_{e' \in E} \mathsf{VP}_{\mathbf{s}}(e')$.

Theorem 2.1 implies that a Nash equilibrium can be verified in polynomial time. A *Covering profile* is a profile $\mathbf{s}$ such that (1) $\mathsf{Support}_{\mathbf{s}}(ep)$ is an Edge Cover of G and (2) $\mathsf{Support}_{\mathbf{s}}(vp)$ is a Vertex Cover of the graph $G(\mathsf{Support}_{\mathbf{s}}(ep))$. It is shown in [12] that a Nash equilibrium $\mathbf{s}$ is a Covering profile. (It is also shown in [12] that a Covering profile is not necessarily a Nash equilibrium.) An *Independent Covering profile* [12] is a uniform attacker symmetric Covering profile $\mathbf{s}$ such that (1) $\mathsf{Support}_{\mathbf{s}}(vp)$ is an Independent Set of G and (2) each vertex in $\mathsf{Support}_{\mathbf{s}}(vp)$ is incident to exactly one edge in $\mathsf{Support}_{\mathbf{s}}(ep)$. Clearly, by the fact that $\mathbf{s}$ is a Covering profile and Condition (3), it follows that for an Independent Covering profile $\mathbf{s}$, $|\mathsf{Support}_{\mathbf{s}}(vp)| = |\mathsf{Support}_{\mathbf{s}}(ep)|$. It is finally shown in [12] that for an Independent Covering profile $\mathbf{s}$, there is a Matching $M \subseteq \mathsf{Support}_{\mathbf{s}}(ep)$ that matches each vertex in $V \backslash \mathsf{Support}_{\mathbf{s}}(vp)$ to some vertex in $\mathsf{Support}_{\mathbf{s}}(vp)$; note that $|M| = |V \backslash \mathsf{Support}_{\mathbf{s}}(vp)|$. In the same work, it was proved that an Independent Covering profile is a Nash equilibrium, called a *Matching NE* [12]. A graph-theoretic characterization of Matching NE is provided there:

Theorem 2.2 ([12]) *A graph G admits a Matching Nash equilibrium if and only if G has an Expanding Independent Set.*

We study algorithmic problems of existence and computation of various classes of Nash equilibria for the considered game.

CLASS NE EXISTENCE	FIND CLASS NE
INSTANCE: A graph $G(V, E)$.	INSTANCE: A graph $G(V, E)$.
QUESTION: Does $\Pi(G)$ admit a CLASS Nash equilibrium?	OUTPUT: A CLASS Nash equilibrium of G.

Variable CLASS takes values GENERAL, MATCHING, PERFECT MATCHING, DEFENDER UNIFORM and ATTACKERS SYMMETRIC UNIFORM; it determines the classes of general, Matching, Perfect Matching, Defender Uniform and Attackers Symmetric, Uniform Nash equilibria, respectively. We note that for all values of CLASS, membership of a profile in CLASS can be verified in polynomial time. Since a Nash equilibrium can be verified in polynomial time (by Theorem 2.1), it follows that CLASS NE EXISTENCE $\in \mathcal{NP}$.

The Price of Defense is the worst-case ratio, over all Nash equilibria $\mathbf{s}$, of $\frac{\nu}{\mathsf{IP}_{\mathbf{s}}(ep)}$.

3 Some Problems from Graph Theory

For our negative results, we will use two known, $\mathcal{NP}$-complete graph-theoretic problems, stated here in the style of Garey and Johnson [6]:

DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS
INPUT: A directed graph $G(V, E)$.
QUESTION: Can the vertices of G be partitioned into disjoint sets $V_1, \cdots V_k$, for some k, such that each V_i contains at least three vertices and induces a subgraph that contains a Hamiltonian circuit?

UNDIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS OF SIZE AT LEAST SIX
INPUT: An undirected graph $G(V, E)$.
QUESTION: Can the vertices of G be partitioned into disjoint sets $V_1, \cdots V_k$, for some k, such that each V_i contains at least six vertices and induces a subgraph that contains a Hamiltonian circuit?

These problems were proved $\mathcal{NP}$-complete in [19] and [17], respectively. Furthermore, for our positive results, we will consider two (to the best of our knowledge) new graph-theoretic problems:

MAXIMUM INDEPENDENT SET EQUAL MINIMUM EDGE COVER
INSTANCE: A graph $G(V, E)$.
OUTPUT: A Maximum Independent Set of G of size $\beta'(G)$ if $\alpha(G) = \beta'(G)$, or NO if $\alpha(G) < \beta'(G)$.

MAXIMUM INDEPENDENT SET EQUAL HALF ORDER
INSTANCE: A graph $G(V, E)$.
OUTPUT: A Maximum Independent Set of G of size $\frac{|V|}{2}$ if $\alpha(G) = \frac{|V|}{2}$, or NO if such does not exist.

For these two new problems, we use reductions to 2SAT (solvable in polynomial time [4]) to prove:

Proposition 3.1 MAXIMUM INDEPENDENT SET EQUAL MINIMUM EDGE COVER $\in \mathcal{P}$

Sketch of Proof. Compute a Minimum Edge Cover EC of G. Recall that EC consists of vertex-disjoint star graphs. Use EC to construct a 2SAT instance ϕ with variable set V as follows:

(1) For each edge $(u, v) \in E$, add the clause $(\bar{u} \vee \bar{v})$ to ϕ.

(2) For each single-edge star graph $(u, v) \in EC$, add the clause $(u \vee v)$ to ϕ.

(3) For each multiple-edge star graph of EC with center vertex u, add the clause $(\bar{u} \vee \bar{u})$ to ϕ.

We prove that G has an Independent Set of size $|EC|$ (hence, $\alpha(G) = \beta'(G)$) if and only if ϕ is satisfiable; when ϕ is satisfiable, the set $\{u \mid \chi(u) = 1\}$ is such a Maximum Independent Set. ■

Similar to Proposition 3.1, we prove:

Proposition 3.2 MAXIMUM INDEPENDENT SET EQUAL HALF ORDER $\in \mathcal{P}$, *when restricted to the class of graphs having a Perfect Matching.*

Sketch of Proof. Compute a Perfect Matching M of G. Use M to construct a 2SAT instance ϕ with variable set V as follows:

(1) For each edge $(u, v) \in E$, add the clause $(\bar{u} \vee \bar{v})$ to ϕ.

(2) For each edge $(u, v) \in M$, add the clause $(u \vee v)$ to ϕ.

We prove that G has $\alpha(G) = \frac{|V|}{2}$ if and only if ϕ is satisfiable; when ϕ is satisfiable, the set $\{u \mid \chi(u) = 1\}$ is such a Maximum Independent Set of size $\frac{|V|}{2}$. ■

4 General Nash Equilibria

Denote as $\hat{\Pi}(G)$ the special case of $\Pi(G)$ with $\nu = 1$; so, $\hat{\Pi}(G)$ is a Two-Players game. Consider a Nash equilibrium $\hat{\mathbf{s}}$ of $\hat{\Pi}(G)$. Constuct from $\hat{\mathbf{s}}$ a vp-symmetric profile $\mathbf{s}$ for $\Pi(G)$ where for each vertex player vp_i, for each vertex $v \in V$, $s_i(v) = \hat{s}_{vp}(v)$, where vp denotes the (single) vertex player of $\hat{\Pi}(G)$; for the edge player ep, for each edge $e \in E$, $s_{ep}(e) = \hat{s}_{ep}(e)$.We prove that $\mathbf{s}$ satisfies the characterization of Nash equilibria in Theorem 2.1 (assuming that $\hat{\mathbf{s}}$ does); so, $\mathbf{s}$ is a Nash equilibrium for $\Pi(G)$. Hence, a Nash equilibrium $\mathbf{s}$ for $\Pi(G)$ can be computed from a Nash equilibrium $\hat{\mathbf{s}}$ for $\hat{\Pi}(G)$ in polynomial time.

We now prove that the two players game $\hat{\Pi}(G)$ is a *constant-sum* (two players) game: for each profile $\hat{\mathbf{s}}$, $\mathsf{IP}_{\hat{\mathbf{s}}}(vp) + \mathsf{IP}_{\hat{\mathbf{s}}}(ep)$ is a constant (independent of $\hat{\mathbf{s}}$). Clearly,

$$\begin{aligned}
\mathsf{IP}_{\hat{\mathbf{s}}}(vp) + \mathsf{IP}_{\hat{\mathbf{s}}}(ep) &= \sum_{v \in V} \hat{s}_{vp}(v) \cdot \left(1 - \sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} \hat{s}_{ep}(e)\right) + \sum_{(u,v)=e\, \in E} \hat{s}_{ep}(e) \cdot (\hat{s}_{vp}(u) + \hat{s}_{vp}(v)) \\
&= \sum_{v \in V} \hat{s}_{vp}(v) - \sum_{v \in V} \hat{s}_{vp}(v) \left(\sum_{e \in \mathsf{Edges}_{\mathbf{s}}(v)} \hat{s}_{ep}(e)\right) + \sum_{(u,v)=e\, \in E} \hat{s}_{ep}(e) \cdot (\hat{s}_{vp}(u) + \hat{s}_{vp}(v)) \\
&= 1 - \sum_{(u,v)=e\, \in E} \hat{s}_{ep}(e) \cdot (\hat{s}_{vp}(u) + \hat{s}_{vp}(v)) + \sum_{(u,v)=e\, \in E} \hat{s}_{ep}(e) \cdot (\hat{s}_{vp}(u) + \hat{s}_{vp}(v)) = 1
\end{aligned}$$

Since a Nash equilibrium of a constant-sum, two-players game can be computed in polynomial time via reduction to Linear Programming [16] (which can be solved in polynomial time [9]), we obtain:

Theorem 4.1 FIND GENERAL NE $\in \mathcal{P}$

5 Matching Nash Equilibria

We first prove some graph-theoretic properties of Matching NE.

Proposition 5.1 *In a Matching NE* $\mathbf{s}$, $\mathsf{Support}_{\mathbf{s}}(vp)$ *is a Maximum Independent Set of* G.

Sketch of Proof. By definition of a Matching NE, $\mathsf{Support}_{\mathbf{s}}(vp)$ is an Independent Set of G, or $V \backslash \mathsf{Support}_{\mathbf{s}}(vp)$ is a Vertex Cover of G. So, it suffices to prove that $V \backslash \mathsf{Support}_{\mathbf{s}}(vp)$ is a Minimum Vertex Cover of G. (Since $\alpha(G) + \beta(G) = |V|$, this will imply that $\mathsf{Support}_{\mathbf{s}}(vp)$ is a Maximum Independent Set of G.) Clearly, $|V \backslash \mathsf{Support}_{\mathbf{s}}(vp)| \geq \beta(G)$. Since $\mathbf{s}$ is an Independent Covering profile, there is a Matching M of G such that $|M| = |V \backslash \mathsf{Support}_{\mathbf{s}}(vp)|$. Thus, $\beta(G) \geq \alpha'(G) \geq |M| = |V \backslash \mathsf{Support}_{\mathbf{s}}(vp)|$. It follows that $|V \backslash \mathsf{Support}_{\mathbf{s}}(vp)| = \beta(G)$, as needed. ■

We continue to prove:

Proposition 5.2 *In a Matching NE* $\mathbf{s}$, $\mathsf{Support}_{\mathbf{s}}(ep)$ *is a Minimum Edge Cover of* G.

Sketch of Proof. Assume, by way of contradiction, that $\mathsf{Support}_{\mathbf{s}}(ep)$ is *not* a Minimum Edge Cover of G. This implies that $|\mathsf{Support}_{\mathbf{s}}(ep)| > \beta'(G)$. Since $\mathbf{s}$ is a Matching NE, $|\mathsf{Support}_{\mathbf{s}}(ep)| = |\mathsf{Support}_{\mathbf{s}}(vp)|$. It follows that $|\mathsf{Support}_{\mathbf{s}}(vp)| > \beta'(G)$. Since $\mathsf{Support}_{\mathbf{s}}(vp)$ is an Independent Set of G, $|\mathsf{Support}_{\mathbf{s}}(vp)| \leq \alpha(G)$. It follows that $\alpha(G) > \beta'(G)$. A contradiction. ■

We are now ready to prove:

Theorem 5.3 *The graph* G *admits a Matching NE if and only if* $\alpha(G) = \beta'(G)$.

Sketch of Proof. Assume first that $\alpha(G) = \beta'(G)$. Let IS and EC be a Maximum Independent Set and a Minimum Edge Cover of G, respectively. So, $|IS| = |EC|$. Consider a uniform, attackers symmetric profile $\mathbf{s}$ with $\mathsf{Support}_{\mathbf{s}}(vp) = IS$ and $\mathsf{Support}_{\mathbf{s}}(ep) = EC$. Thus, $|\mathsf{Support}_{\mathbf{s}}(vp)| = |\mathsf{Support}_{\mathbf{s}}(ep)|$. We will prove that $\mathbf{s}$ is an Independent Covering profile.

By construction, $\mathbf{s}$ is uniform and attackers symmetric, $\mathsf{Support}_{\mathbf{s}}(ep)$ is an Edge Cover of G and $\mathsf{Support}_{\mathbf{s}}(vp)$ is an Independent Set of G. So, there remains to show Condition (2) in the definition of a Covering profile and additional Condition (2) in the definition of an Independent Covering profile. Since EC is a *Minimum* Edge Cover, it is a union of disjoint star graphs. Since $|\mathsf{Support}_{\mathbf{s}}(vp)| = |\mathsf{Support}_{\mathbf{s}}(ep)|$ and $\mathsf{Support}_{\mathbf{s}}(vp)$ is an Independent Set of G, it follows that $\mathsf{Support}_{\mathbf{s}}(vp)$ consists of *all* terminal vertices of the star graphs. This implies that both *(i)* $\mathsf{Support}_{\mathbf{s}}(vp)$ is a Vertex Cover of the graph $G(\mathsf{Support}_{\mathbf{s}}(ep))$ (Condition (2)) and *(ii)* each vertex in $\mathsf{Support}_{\mathbf{s}}(vp)$ is incident to exactly one edge of $\mathsf{Support}_{\mathbf{s}}(ep)$ (additional Condition (2)). Hence, $\mathbf{s}$ is an Independent Covering profile. Since an Independent Covering profile is a Nash equilibrium, the claim follows.

Assume now that G admits a Matching NE $\mathbf{s}$. By Proposition 5.1, $\mathsf{Support}_{\mathbf{s}}(vp)$ is a Maximum Independent Set of G, so that $|\mathsf{Support}_{\mathbf{s}}(vp)| = \alpha(G)$. By Proposition 5.2, $\mathsf{Support}_{\mathbf{s}}(ep)$ is a Minimum Edge Cover of G, so that $|\mathsf{Support}_{\mathbf{s}}(ep)| = \beta'(G)$. Since $\mathbf{s}$ is a Matching NE, $|\mathsf{Support}_{\mathbf{s}}(vp)| = |\mathsf{Support}_{\mathbf{s}}(ep)|$. It follows that $\alpha(G) = \beta'(G)$, as needed. ■

The constructive parts of the sufficiency proofs of Proposition 3.1 and Theorem 5.3 yield together a polynomial time algorithm $\mathsf{MatchingNE}$ to compute a Matching NE, if one exists:

Algorithm MatchingNE
INPUT: A graph $G(V, E)$.
OUTPUT: The supports in a Matching NE $\mathbf{s}$ for G, or NO if such does not exist.

1. Compute a Minimum Edge Cover EC of G.
2. Construct an instance ϕ of 2SAT as follows:
    - For each edge $(u, v) \in E$, add the clause $(\bar{u} \vee \bar{v})$ to ϕ.
    - For each single-edge star graph $(u, v) \in EC$, add the clause $(u \vee v)$ to ϕ.
    - For each multiple-edge star graph of EC with center vertex u, add the clause $(\bar{u} \vee \bar{u})$ to ϕ.
3. Compute a satisfying assignment χ of ϕ, or output NO if such does not exist.
4. Set $IS = \{u \mid \chi(u) = 1\}$.
5. Set $\mathsf{Support}_{\mathbf{s}}(ep) := EC$ and $\mathsf{Support}_{\mathbf{s}}(vp) := IS$.

Theorem 5.4 *Algorithm* MatchingNE *solves* FIND MATCHING NE *in time* $O\left(\sqrt{|V|}|E| \cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$.

We finally prove:

Proposition 5.5 *In a Matching NE, the Price of Defense is* $\alpha(G)$.

Sketch of Proof. Fix a Matching NE $\mathbf{s}$. By the fact that $\mathbf{s}$ is is a Covering profile and Condition (1) of a Matching NE, it follows that *exactly one* of the endpoints (say v) of edge e is contained in $\mathsf{Support}_{\mathbf{s}}(vp)$. By Lemma 5.1, $|\mathsf{Support}_{\mathbf{s}}(vp)| = \alpha(G)$. Since $\mathbf{s}$ is a NE, the Price of Defense is $\frac{\nu}{\mathsf{IP}_{\mathbf{s}}(ep)} = \frac{\nu}{\mathsf{VP}_{\mathbf{s}}(e)} = \frac{\nu}{\mathsf{VP}_{\mathbf{s}}(v)} = \frac{\nu}{\frac{\nu}{|\mathsf{Support}_{\mathbf{s}}(vp)|}} = \alpha(G)$, as needed. ■

6 Perfect Matching Nash Equilibria

A *Perfect Matching NE* is a Matching NE $\mathbf{s}$ such that $\mathsf{Support}_{\mathbf{s}}(ep)$ is a Perfect Matching of G. For the characterization, we first prove:

Proposition 6.1 *For a Perfect Matching NE* $\mathbf{s}$, $|\mathsf{Support}_{\mathbf{s}}(vp)| = \frac{|V|}{2}$.

Sketch of Proof. By definition of a Perfect Matching NE, $\mathsf{Support}_{\mathbf{s}}(ep)$ is a Perfect Matching of G; so, $\mathsf{Support}_{\mathbf{s}}(ep) = \frac{|V|}{2}$. Since a Perfect Matching NE is a special case of a Matching NE (that is, an Independent Covering profile), $|\mathsf{Support}_{\mathbf{s}}(vp)| = |\mathsf{Support}_{\mathbf{s}}(ep)|$. It follows that $|\mathsf{Support}_{\mathbf{s}}(vp)| = \frac{|V|}{2}$, as needed. ■

We are now ready to prove a characterization of Perfect Matching NE:

Theorem 6.2 *A graph G admits a Perfect Matching NE if and only if G has a Perfect Matching and* $\alpha(G) = \frac{|V|}{2}$.

Sketch of Proof. Assume first that G has a Perfect Matching M and $\alpha(G) = \frac{|V|}{2}$. Consider a Maximum Independent Set IS of G. Define a uniform, attackers symmetric profile $\mathbf{s}$ by setting $\mathsf{Support}_{\mathbf{s}}(ep) := M$, $\mathsf{Support}_{\mathbf{s}}(vp) := IS$. By the choice of $\mathsf{Support}_{\mathbf{s}}(ep)$, we only need to prove that $\mathbf{s}$ is an Independent Covering profile. Since a Perfect Matching is a Minimum Edge Cover, this reduces to the corresponding proof of Theorem 5.3 (where $\mathsf{Support}_{\mathbf{s}}(ep)$ was chosen as a Minimum Edge Cover).

Assume now that G admits a Perfect Matching NE $\mathbf{s}$. By definition of Perfect Matching NE, $\mathsf{Support}_{\mathbf{s}}(ep)$ is a Perfect Matching of G. Since a Perfect Matching NE is a special case of a Matching NE, Proposition 5.1 applies to yield that $\mathsf{Support}_{\mathbf{s}}(vp) = \alpha(G)$. By Proposition 6.1, $\mathsf{Support}_{\mathbf{s}}(vp) = \frac{|V|}{2}$. It follows that $\alpha(G) = \frac{|V|}{2}$, as needed. ■

The constructive parts of the sufficiency proof of Proposition 3.2 and Theorem 6.2 yield together a polynomial time algorithm $\mathsf{PerfectMatchingNE}$ to compute a Perfect Matching NE, if one exists.

Algorithm PerfectMatchingNE
INPUT: A graph $G(V, E)$.
OUTPUT: The supports in a Perfect Matching NE $\mathbf{s}$ for G, or NO if such does not exist.

1. Compute a Perfect Matching M of G, or output NO if such does nots exist.
2. Construct an instance ϕ of 2SAT as follows:
   - For each edge $(u, v) \in E$, add the clause $(\bar{u} \vee \bar{v})$ to ϕ.
   - For each edge $(u, v) \in M$, add the clause $(u \vee v)$ to ϕ.
3. Compute a satisfying assignment χ of ϕ, or output NO if such does not exist.
4. Set $IS = \{u \mid \chi(u) = 1\}$.
5. Set $\mathsf{Support}_{\mathbf{s}}(ep) := M$ and $\mathsf{Support}_{\mathbf{s}}(vp) := IS$.

Theorem 6.3 *Algorithm* PerfectMatchingNE *solves* FIND PERFECT MATCHING NE *in time* $O\left(\sqrt{|V|}|E| \cdot \log_{|V|} \frac{|V|^2}{|E|}\right)$.

Observe that a Perfect Matching NE is a Matching NE for which, by Theorem 6.2, $\alpha(G) = \frac{|V|}{2}$. Hence, Proposition 5.5 implies:

Theorem 6.4 *In a Perfect Matching Nash equilibrium, the Price of Defense is* $\frac{|V|}{2}$.

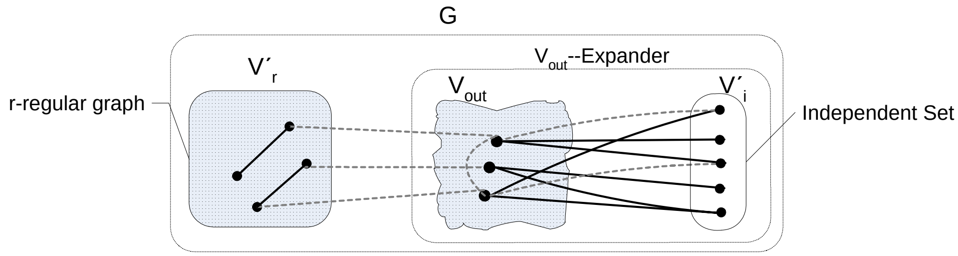

Figure 1: An illustration of the characterization of a Defender Uniform Nash equilibrium for G.

7 Defender Uniform Nash Equilibria

A *Defender Uniform Nash equilibrium* is a Defender Uniform profile that is a Nash equilibrium. We prove a characterization of graphs admitting Defender Uniform Nash equilibria:

Theorem 7.1 *A graph G admits a Defender Uniform Nash equilibrium if and only if there are sets $V' \subseteq V$ and $E' \subseteq E$ such that:*

1. (i) $\forall\, v \in V'$, $d_{G(E')}(v) = r$, *for some integer* r, $1 \leq r \leq \delta(G)$, *and* (ii) $\forall\, v \in V \backslash V'$, $r \leq d_{G(E')}(v)$.
2. V' *can be partitioned into two disjoint sets* V'_i *and* V'_r *such that,*
   (a) V'_i *is an Independent Set of* G,
   (b) *Let* $V_{out} = V \backslash (V'_i \cup V'_r)$. *The graph* $G(E'(V'_i \cup V_{out})))$ *is a* $\{V'_i, V_{out}\}$*-bipartite graph and a* V_{out}*-Expander graph.*
   (c) *It holds that* $E'(V'_r) \cup E'(V'_i \cup V_{out}) = E'$.

An illustration of a Defender Uniform Nash equilibrium for a graph G is shown in Figure 1. The edges of $G(E')$ are shown with dark lines, the rest edges of G are shown with either dotted edges or with a shadow in corresponding subgraphs of G. We now turn to studying the computational complexity of DEFENDER UNIFORM NE EXISTENCE. We prove:

Theorem 7.2 DEFENDER UNIFORM NE EXISTENCE *is* $\mathcal{NP}$*-complete.*

Proof. Recall that DEFENDER UNIFORM NE EXISTENCE $\in \mathcal{NP}$. To prove $\mathcal{NP}$-hardness, we reduce from DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS. From any instance of this problem, $G_d(V_d, A_d)$, we first construct an undirected graph $G(V, E)$ as follows:

1. Set $G(V, E)$ to be an empty graph.
2. Add to G two cycle graphs of size 5; denoted by C_1 and C_2. Set $V(C_i) = \{v_i^k : 1 \leq k \leq 5\}, 1 \leq i \leq 2$.
3. From $G_d(V_d, A_d)$, compute an undirected graph G^1 as follows:
   - For each $v \in V_d$, add vertices v^1, v^2, v^3 to $V(G^1)$ and edges $(v^1, v^2)), (v^2, v^3)$ to $E(G^1)$.
   - For each edge $(v, u) \in A_d$ do the following:
     - If edge $(u, v) \notin A_d$ add edge (v^3, u^1) to $E(G^1)$.
     - Otherwise (edge $(u, v) \in A_d$) add the following to G^1:
       * Add vertices vu^4, e_{vu}, e'_{vu}, e_{uv}, e'_{uv},uv^4, called *vu-vertices*, to $V(G^1)$.
       * Add edges (v^1, vu^4), (v^3, vu^4), (vu^4, e_{vu}), (vu^4, e'_{uv}), (e'_{uv}, e_{uv}), (uv^4, uv^4), (u^3, uv^4), (uv^4, e_{uv}), (uv^4, e'_{vu}), (e'_{vu}, e_{vu}) and (e_{uv}, e_{vu}) to $E(G^1)$.
4. Add $V(G^1)$ to $V(G)$ and $E(G^1)$ to $E(G)$.
5. For each $v \in V(G^1)$, add edges (v_1^1, v) to $E(G)$.

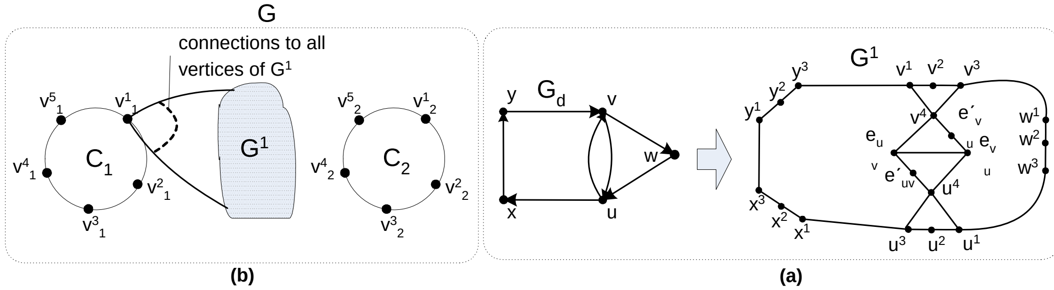

Figure 2: An illustration of (a) the transformation of a directed graph G_d to an undirected graph G^1 and (b) the construction of the (undirected) graph G.

Figure 2 presents an example of (a) the transformation of a directed graph G_d to the undirected graph G^1 and (b) the construction of the (undirected) graph G from graphs G^1, C_1 and C_2. In the following, we adapt the notation of Theorem 7.1. Our analysis will establish several structural properties of Defender Uniform Nash equilibria. We first prove:

Lemma 7.3 *In a Defender Uniform Nash equilibrium,* (i) $r = 2$, (ii) $C_2 \subseteq G(E')$, (iii) $V(C_2) \subseteq V'_r$, (iv) $V(C_1) \subseteq V'_r$, (v) $C_1 \subseteq G(E'(V'_r))$ *and* (vi) $\mathsf{Neigh}_{G(E)}(v_1^1) = \{v_1^2, v_1^5\}$.

Lemma 7.4 *In a Defender Uniform Nash equilibrium of G, $V(G^1) \subseteq V'_r$.*

We are now ready to prove a reduction from DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS (with instance G_d) to UNDIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS OF SIZE AT LEAST SIX (with instance G^1).

Lemma 7.5 *If* DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS *is answered positively for G_d, then* UNDIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS OF SIZE AT LEAST SIX *is answered positively for G^1.*

Lemma 7.6 *If* UNDIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS OF SIZE AT LEAST SIX *is answered positively for G^1 then* DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS *problem is answered positively for G_d.*

We finally proceed to the details of our $\mathcal{NP}$-hardness proof. We prove:

Lemma 7.7 *If* DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS *is answered positively for G_d, then G admits a Defender Uniform Nash equilibrium.*

Lemma 7.8 *If G admits a Defender Uniform Nash equilibrium, then* DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS *is answered positively for G_d.*

Concluding the proof are Lemmas 7.7 and 7.8. (Lemmas 7.5 and 7.6 provide in isolation an alternative $\mathcal{NP}$-completeness proof to one in [17] for UNDIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS.) ■

We finally prove:

Theorem 7.9 *In a Defender Uniform Nash equilibrium, the Price of Defense is $|V'_i| + \frac{|V'_r|}{2}$, where the notation refers to Theorem 7.1.*

Sketch of Proof. In the proof of the characterization of Defender Uniform Nash equilibria, we showed that $\mathsf{IP}_{\mathbf{s}}(ep) = \dfrac{\nu}{|V_i'| + \frac{|V_r'|}{2}}$. Thus, the Price of Defense is $\frac{\nu}{\mathsf{IP}_{\mathbf{s}}(ep)} = |V_i'| + \frac{|V_r'|}{2}$. ■

The worst case (maximum value) of the Price of Defense in Theorem 7.9 is obtained when $\mathsf{Support}_{\mathbf{s}}(vp)$ is the *maximum* possible and set V_r' is empty. Then, the Defender Uniform Nash equilibrium $\mathbf{s}$ is actually a Matching NE and, by Proposition 5.5, the Price of Defense is equal to $\alpha(G)$. For the best case (minimum value), recall that by Condition 2(b) of the characterization of Defender Uniform Nash equilibria, the graph $G(E'((V_i' \cup V_{out}))$ is a V_{out}-Expander graph. Thus, by definition of Expander graphs, $|V_{out}| \leq |V_i'|$. Since $|V| = |V_r'| + |V_i'| + |V_{out}|$, it follows that $|V_r'| + 2|V_i'| \geq |V|$. Hence, $|V_i'| + \frac{|V_r'|}{2} \geq \frac{|V| - |V_r'|}{2} = \frac{|V|}{2}$. So, Defender Uniform Nash equilibria fall between Perfect Matching and Matching Nash equilibria (with respect to Price of Defense), for the case where $\frac{|V|}{2} \leq \alpha(G)$.

8 Attacker Symmetric Uniform Nash Equilibria

An *Attacker Symmetric Uniform NE* is an Attacker Symmetric Uniform profile that is a NE. We prove a characterization of graphs admitting Attacker Symmetric Uniform Nash equilibria:

Theorem 8.1 *A graph G admits an Attacker Symmetric Uniform Nash equilibrium if and only if:*

(1) *There is a probability distribution $p : E \rightarrow [0,1]$ such that* (1/a) $\sum_{e \in \mathsf{Edges}_G(v)} p(e) = \sum_{e' \in \mathsf{Edges}_G(v')} p(e')$, $\forall\, v, v' \in V$ *and* (1/b) $\sum_{e \in \mathsf{Edges}_G(v)} p(e) > 0 \;\forall\, v \in V$, **OR**

(2) $\alpha(G) = \beta'(G)$.

We are now ready to prove:

Theorem 8.2 FIND ATTACKERS SYMMETRIC UNIFORM NE $\in \mathcal{P}$

Sketch of Proof. We prove that Conditions (1) and (2) can be checked in polynomial time. Condition (1) can checked in polynomial time via solving a linear system. Condition (2) was considered before for Matching Nash equilibria (see Theorem 5.3 and Proposition 3.1). ■

Finally we show:

Theorem 8.3 *In an Attacker Symmetric Uniform NE, the Price of Defense is either $\frac{|V|}{2}$ or $\alpha(G)$.*

Proof. Fix an Attacker Symmetric Uniform NE $\mathbf{s}$. We compute the Price of Defense for each one of Conditions (1) and (2) in Theorem 8.1. For Condition (2), note that $\mathsf{Support}_{\mathbf{s}}(vp) = |V|$. Since $\mathbf{s}$ is a attackers symmetric, vertex players uniform profile, it follows that $\mathsf{VP}_{\mathbf{s}}(v) = \sum_{vp_i \in \mathcal{N}_{vp}} \frac{1}{|\mathsf{Support}_{\mathbf{s}}(vp)|} = \frac{\nu}{|V|}$, for any $v \in \mathsf{Support}_{\mathbf{s}}(vp)$. Furthermore, since $\mathbf{s}$ is a NE, it follows that $\mathsf{IP}_{\mathbf{s}}(ep) = \mathsf{VP}_{\mathbf{s}}(e) = \frac{2\nu}{|V|}$, for any $e \in \mathsf{Support}_{\mathbf{s}}(ep)$. Thus, the Price of Defense is $\frac{\nu}{\mathsf{IP}_{\mathbf{s}}(ep)} = \frac{|V|}{2}$. For Condition (2), the case reduces to a Matching NE for which, by Proposition 5.5, the Price of Defense is $\alpha(G)$. ■

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A Graph Theory Notation (Leftovers)

We include here some notation used in the proofs of the Appendix. For a vertex set $U \subseteq V$, denote $\mathsf{Neigh}_G(U) = \{u \notin U : (u,v) \in E$ for some vertex $v \in U\}$. The graph G is *bipartite* if $V = V_1 \cup V_2$ for some disjoint vertex sets $V_1, V_2 \subseteq V$ so that for each edge $(u,v) \in E$, $u \in V_1$ and $v_\in V_2$. Call G a (V_1, V_2)-bipartite graph.

B Proof of Proposition 3.1

Denote as C_1, and C_2 and C_3 the set of clauses of ϕ obtained by rules 1, 2 and 3, respectively. Assume first that G has an Independent Set IS of size $\beta'(G)$. For each variable v of ϕ, set $\chi(v) := 1$ if $v \in IS$ and $\chi(v) := 0$, otherwise. For each edge $(u,v) \in E$ at most one of its vertices can be included in IS. Thus, the set C_1 is satisfied in χ. Since $|IS| = |EC|$, set IS must consists of exactly one vertex of each single-edge star graph and all terminal vertices of each multi-edge star graph of EC. Thus, sets C_2 and C_3 are satifisfable in χ.

Assume now that χ is a satisfying assignment of ϕ. Define set $IS := \{u | \chi(u) = 1\}$. Since for every edge $(u,v) \in EC \subseteq E$, the clauses $(u \vee v)$ and $(\bar{u} \vee \bar{v})$ are both in ϕ. Thus, exactly one of vertices u and v is included in IS. Since all clauses of set C_3 are satisfied in ϕ, all center vertices of EC are excluded from IS. It follows that IS consists of all terminal vertices of EC. Thus, $|IS| = |EC|$. Now, consider any pair of vertices $u, v \in IS$. Then, variables u and v are true under the satisfying assignment, and therefore the clause $\bar{u} \vee \bar{v}$ does not appear in ϕ. Thus, $(u,v) \notin E$, which implies that IS is an Independent Set.

C Proof of Proposition 3.2

Denote as C_1 and C_2 the set of clauses of ϕ obtained by rules 1 and 2, respectively. Assume first that G contains a Maximum Independent Set IS of size $\frac{|V|}{2}$. For each variable v of ϕ, set $\chi(v) := 1$ if $v \in IS$ and $\chi(v) := 0$, otherwise. For every edge $(u,v) \in M$, exactly one of u and v are contained in IS, for otherwise it is impossible to get a Maximum Independent Set with size $\frac{|V|}{2}$. Hence, χ satisfies all clauses of set C_2. Also, for every edge $(u,v) \in E$, at most one of u and v is contained in IS. Thus, χ satisfies all clauses of set C_1. Overall, χ satisfies ϕ.

Assume now that χ is a satisfiable assignment of ϕ. Define set $IS := \{u | \chi(u) = 1\}$. Since for every edge $(u,v) \in M \subseteq E$, the clauses $(u \vee v)$ and $(\bar{u} \vee \bar{v})$ are both in ϕ, it follows that the satisfying assignment makes exactly $\frac{|V|}{2}$ variables true, and thus $|IS| = \frac{|V|}{2}$. Now, consider any pair of vertices $u, v \in IS$. Then, variables u and v are true under the satisfying assignment, and therefore the clause $\bar{u} \vee \bar{v}$ does not appear in ϕ. Thus, $(u,v) \notin E$, which implies that IS is an Independent Set. Since G contains a Perfect Matching it follows that $\alpha(G) \leq |M| = \frac{|V|}{2}$. It follows that IS is a Maximum Independent Set of G and $|IS| = \alpha(G) = \frac{|V|}{2}$.

D Proof of Theorem 7.1

We show the Theorem via the following two Lemmas.

Lemma D.1 *If conditions* (1) *and* (2) *hold then* $\Pi(G)$ *admits a Defender Uniform Nash equilibrium.*

Proof. Construct a Defender Uniform, Attackers Symmetric profile $\mathbf{s}$ with the following supports for the players: $\mathsf{Support}_{\mathbf{s}}(vp) := V'$ and $\mathsf{Support}_{\mathbf{s}}(ep) := E'$. For any $vp_i \in \mathcal{N}_{vp}$, set $s_i(v) := \dfrac{2}{2 \cdot |V_i'| + |V_r'|}$

for all $v \in V_i'$ and $s_i(v) := \frac{1}{2 \cdot |V_i'| + |V_r'|}$ for all $v \in V_r'$. Note that $\mathbf{s}$ is a valid profile of $\Pi(G)$: For any vertex player vp_i, $\sum_{v \in \mathsf{Support_s}(i)} s_i(v) = \sum_{v \in V_i'} \frac{2}{2 \cdot |V_i'| + |V_r'|} + \sum_{v \in V_r'} \frac{1}{2 \cdot |V_i'| + |V_r'|} = \frac{2|V_i'|}{2 \cdot |V_i'| + |V_r'|} + \frac{|V_r'|}{2 \cdot |V_i'| + |V_r'|} = 1$. We next show that $\mathbf{s}$ is a NE.

We first show that profile $\mathbf{s}$ is a Covering profile: By Condition (1), $\mathsf{Support}_\mathbf{s}(ep)$ is an Edge Cover of G. Set $V_r' \subseteq \mathsf{Support}_\mathbf{s}(vp)$ is a Vertex Cover of the edge set $E'(V_r') \subseteq \mathsf{Support}_\mathbf{s}(ep)$ and set $V_i' \subseteq \mathsf{Support}_\mathbf{s}(vp)$ is a Vertex Cover of the edge set $E'(V_i' \cup V_{out}) \subseteq \mathsf{Support}_\mathbf{s}(ep)$. Since $\mathsf{Support}_\mathbf{s}(ep) = E'(V_r') \cup E'(V_i' \cup V_{out})$, it follows that $\mathsf{Support}_\mathbf{s}(vp)$ is a Vertex Cover of the graph $G(\mathsf{Support}_\mathbf{s}(ep))$, which concludes the claim.

Claim D.2 *For any $v \in \mathsf{Support}_\mathbf{s}(vp)$, $P_\mathbf{s}(\mathsf{Hit}(v)) = r \cdot \frac{1}{|E'|} = \min_e P_\mathbf{s}(\mathsf{Hit}(v))$.*

Proof. Since $\mathbf{s}$ is a defender uniform profile, $P_\mathbf{s}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges_s}} s_{ep}(e) = |\mathsf{Edges_s}(v)| = r\ cdot \frac{1}{|E'|}$, when $v \in \mathsf{Support}_\mathbf{s}(vp)$ and $P_\mathbf{s}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges_s}(v)} s_{ep}(e) = |\mathsf{Edges_s}(v)| \geq r \cdot \frac{1}{|E'|}$, when $v \notin \mathsf{Support}_\mathbf{s}(vp)$. The claim follows. ■

Claim D.3 *For any edge $e \in \mathsf{Support}_\mathbf{s}(ep)$, $\mathsf{VP}_\mathbf{s}(e) = \frac{2\nu}{2 \cdot |V_i'| + |V_r'|} = \max_e \mathsf{VP}_\mathbf{s}(e)$.*

Proof. Consider any $e = (u, v) \in E$. We consider all possible cases:
(i) when both $u, v \in V_r'$, by construction, $\mathsf{VP}_\mathbf{s}(e) = \sum_{vp_i \in \mathcal{N}_{vp}} s_i(v) + s_i(u) = \frac{\nu}{2 \cdot |V_i'| + |V_r'|} + \frac{\nu}{2 \cdot |V_i'| + |V_r'|} = \frac{2\nu}{2 \cdot |V_i'| + |V_r'|}$.
(ii) when $v \in V_i'$ and $u \in V_{out}$, then by construction, $\mathsf{VP}_\mathbf{s}(e) = \sum_{vp_i \in \mathcal{N}_{vp}} s_i(v) + s_i(u) = \sum_{vp_i \in \mathcal{N}_{vp}} \frac{2}{2 \cdot |V_i'| + |V_r'|} + 0 = \frac{2\nu}{2 \cdot |V_i'| + |V_r'|}$.
(iii) when $v, u \in V_{out}$, then by construction, $\mathsf{VP}_\mathbf{s}(e) = 0$.

Note that the following cases are not possible: (iv) case $v, u \in V_i'$ can not exists by Condition 2(a). (v) case $v \in V_r'$ and $u \in V_{out} \cup V_i'$ can not exists by Condition 2(d).

Now consider any $e = (u, v) \in E'$. By Condition (2) of Theorem 7.1, only cases (i) or (ii) are possible for e, which concludes that for any $e \in \mathsf{Support}_\mathbf{s}(ep)$, $\mathsf{VP}_\mathbf{s}(e) = \frac{2\nu}{2 \cdot |V_i'| + |V_r'|} = \max_e \mathsf{VP}_\mathbf{s}(e)$.

■

We now returning in the proof of Lemma D.1 to show that $\mathbf{s}$ is a NE. We prove this by showing that $\mathbf{s}$ satisfies the conditions of Theorem 2.1. Profile $\mathbf{s}$ was shown to be a Coverign profile. Moreover, Condition 1 of Theorem 2.1 is satisfied by Claim D.3 and Conditon 2 of the Theorem by Claim D.2. It follows that $\mathbf{s}$ is a NE. ■

Lemma D.4 *A Defender Uniform NE $\mathbf{s}$ satisfies conditions* (1) *and* (2).

Proof. Set $V' = \mathsf{Support}_\mathbf{s}(vp)$ and $E' = \mathsf{Support}_\mathbf{s}(ep)$. We prove that sets V' and E' satisfy the requirements of corresponding sets of Theorem 7.1.

Condition (1): For any veretx of V, since $\mathbf{s}$ is a uniform profile, $P_\mathbf{s}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges_s}(v)} P_\mathbf{s}(ep, e) = \sum_{e \in \mathsf{Edges_s}(v)} \frac{1}{|E'|} = d_{G(E')}(v) \cdot \frac{1}{|E'|}$. Also, since $\mathbf{s}$ is a NE, for any $v \in V'$, $P_\mathbf{s}(\mathsf{Hit}(v) = \min_{v' \in V} P_\mathbf{s}(\mathsf{Hit}(v))$. Thus, $d_{G(E')}(v) = d_{G(E')}(u) = \Delta_{G(E')(V')} = r \leq d_{G(E')}(v')$, for any three vertices $v, u \in V'$, $v' \in V \backslash V'$.

Now consider the vertex of minimum degree in G. By the above results we get that $r \geq d_{G(E')}(v)$. In order this to be feasible it must be that $r \leq \delta_G$. Condition (1) of Theorem 7.1 follows.

Condition (2): Let $V_i' \subseteq V'$ such that V_i' is an Independent Set of G and for any $v \in V_i'$, for each vertex u such that $u \in \mathsf{Neigh}_{G(E')}(v)$ it holds that $u \notin V'$. Let also $V_r' = V' \backslash V_i'$ and $V_{out} = V \backslash V'$.

Condition 2(a): By the definition of set V_i'.
The following Claim will be utilized to prove the rest of the conditions.

Claim D.5 *For any $v \in V_i'$ and $(u, v) \in E'$, it holds that $\mathsf{VP}_{\mathbf{s}}(v) = \max_e \mathsf{VP}_{\mathbf{s}}(e)$ and $u \notin V'$. For any $v \in V_r'$ and $(u, v) \in E'$, it holds that $\mathsf{VP}_{\mathbf{s}}(v) < \max_e \mathsf{VP}_{\mathbf{s}}(e)$ and $u \in V_r'$.*

Proof. Consider any $v \in V_i'$ and any edge $(v, u) \in E'$ (such edge exists because V' is a Vertex Cover of E', since $\mathbf{s}$ is a Covering profile). By definition of set V_i', $u \notin V'$. Since $\mathbf{s}$ is a NE, it follows that $\mathsf{VP}_{\mathbf{s}}(e) = \mathsf{VP}_{\mathbf{s}}(v) = \max_e \mathsf{VP}_{\mathbf{s}}(e)$.

Consider any $v \in V_r'$ and any edge $(v, u) \in E'$ (such an edge exists, similarly to above). By the definition of set V_r', it holds that also $u \in V_r'$. Since $\mathbf{s}$ is a NE, it follows that $\mathsf{VP}_{\mathbf{s}}(v), \mathsf{VP}_{\mathbf{s}}(u) < \max_e \mathsf{VP}_{\mathbf{s}}(e)$. ■

Condition 2(c): Consider any $e = (u, v) \in E'$. If $v \in V_i'$ then by Claim D.5, $u \in V_{out}$ and thus e is contained in the bipartite graph $E'(V_i' \cup V_{out})$. If $v \in V_r'$, then by the same claim, $u \in V_r'$ and hence e is contained in $E'(V_r')$. Finally, note the remaining case, where both $v, u \notin V_{out}$ is not possible since V' is a Vertex Cover of E' ($\mathbf{s}$ is a overing profile). Thus, any edge of E' is contained either in $E'(V_i' \cup V_{out})$ or in $E'(V_r')$.

Condition 2(b): Consider any edge $e = (u, v) \in E'$ such that $v \in V_r'$. By Claim D.5, also $u \in V_r'$. This combined with that $\Delta_{G(E')}(v) = r$, for any $v \in V'$ proved in Condition (1) above, concludes that $G(E'(V_r'))$ is an r-regular graph.

Consider any $U \subseteq V_{out}$. By Condition (1) showed above, $d_{G(E')}(u) \geq r$, for any $u \in U$. Moreover by 3(d), all neighbors of each such u, i.e., set $\mathsf{Neigh}_{G(E')}(U)$, are included in V_i'. This combined with that for any $v \in V_i' \subseteq V'$, $d_{G(E')}(v) = r$ (proved in (2) above), concludes that it must be that $|\mathsf{Neigh}_{G(E')}(U)| \geq r \cdot |U|$. Thus, $|\mathsf{Neigh}_{G(E')}(U)| \geq |U|$ and $G(E'(V_i' \cup V_{out}))$ is a V_{out}-expander graph.

Consider any $v, u \in V_i'$. Set V_i' is an Independent Set of G by its definition. Thus, $(v, u) \notin E'$. Also, for any $v, u \in V_{out}$, there is no edge $e = (v, u) \in E'$, since otherwise $\mathsf{VP}_{\mathbf{s}}(e) = 0 < \max \mathsf{VP}_{\mathbf{s}}(e)$, contradiction to that $\mathsf{VP}_{\mathbf{s}}(e) = \max \mathsf{VP}_{\mathbf{s}}(e)$, for any $e \in E'$ in a NE. The above two observations conclude that $G(E'(V_i' \cup V_{out}))$ is a $\{V_i', V_{out}\}$-bipartite graph. ■

E Proof of Theorem 7.2

E.1 Proof of Lemma 7.3

Note that $\delta_G = 2$, since the vertices of C_2 are connected to no other vertex of G besides the vertices of C_2. Thus, by Condition (1) of Theorem 7.1, $r \leq \delta_G = 2$, as required for condition (i).

Recall that C_2 is disconnected from the rest of the graph G. Thus, we can apply Theorem 7.1 on this graph. Observe that the graph can not be partitioned into two sets IS, VC such that IS is an Independent Set of G and C_2 is a VC-expander. So, by Theorem 7.1 on C_2, in any Defender Uniform Nash equilibrium of $\Pi(C_2)$ the support of the edge player must be an r'-regular graph, with $1 \leq r' \leq 2$. Since C_2 does not have a Perfect Matching it must be that $r' = 2$, which determines parameter r of G to be equal to 2. The above fundings conclude also that $C_2 \subseteq G(E')$and that $V(C_2) \subseteq V_r'$, as required for conditions (ii) and (iii), respectively.

We next show some auxiliary Claims.

Claim E.1 *In any Defender Uniform Nash equilibrium of G, $C_1 \subseteq G(E')$.*

Proof. By Lemma 7.3, for any any $v_1^j \in V(C_1)$, $1 \leq i \leq 5$, it must be $d_{G(E')}(v_1^j) \geq 2$ and since $d_G(v_1^j) = 2$, we get that $d_{G(E')}(v_1^j) = 2$ and that $C_1 \subseteq G(E')$. ■

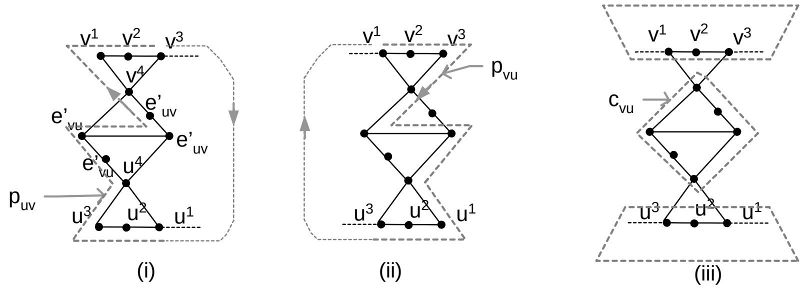

Figure 3: The possible vertex disjoint cycles covering of graph G^1.

Claim E.2 *In any Defender Uniform Nash equilibrium of G, $v_1^1 \in V'$.*

Proof. Assume the contrary. By Claim E.1, $C_1 \subseteq G(E')$. By the characterization of a NE, V' is a Vertex Cover of E'. So, in order V' to cover the edge set C_1, it should include some vertices of the set $v_1^2, \cdots, v_1^5$. This vertex set will unavoidably contain at least two neighbour vertices in C_1. Moreover, two of these vertices would be neigbours to v_1^1, since $v_1^1 \notin V'$ and hence it must be that $d_{E'}(v_1^1) \geq 2$. So, assume w.l.g. that these vertices include v_1^2, v_1^3, v_1^5. But then, $\mathsf{VP}_{\mathbf{s}}(v_1^2, v_1^3)) = \mathsf{VP}_{\mathbf{s}}(v_1^2) + \mathsf{VP}_{\mathbf{s}} v_1^3) > \mathsf{VP}_{\mathbf{s}}(v_1^1, v_1^2)) = \mathsf{VP}_{\mathbf{s}}(v_1^2)$. This contradicts to Condition 2 of the characterization of a NE for edges (v_1^2, v_1^3), (v_1^1, v_1^2), contained in E'. ■

By Claim E.2, $v_1^1 \in V'$. We first show that $v_1^1 \notin V_i'$. Assume the contrary. Then, since V_i' is an Independent Set of G, vertices v_1^2, v_1^5, which are neighbours to v_1^1 in G are not included in V'. Recall that $C_1 \subseteq G(E')$ (Claim E.1) and that and V' is a Vertex Cover of $G(E')$ ($\mathbf{s}$ is a Covering profile). Thus, in order V' to cover edge set C_1, it should include both vertices of set $\{v_1^3, v_1^4\}$. But then $\mathsf{VP}_{\mathbf{s}}(v_1^3, v_1^4)) = \mathsf{VP}_{\mathbf{s}}(v_1^3) + \mathsf{VP}_{\mathbf{s}} v_1^4) > \mathsf{VP}_{\mathbf{s}}(v_1^4, v_1^5)) = \mathsf{VP}_{\mathbf{s}}(v_1^4)$. This contradicts to Condition 2 of the characterization of a NE for edges $(v_1^3, v_1^4), (v_1^4, v_1^5)$, contained in E'. Thus, $v_1^1 \in V_r'$.

We next show that the rest of the vertices of C_2 are also included in V'. Recall that V' is a Vertex Cover of E' and that $C_1 \subseteq G(E')$. In order V' to cover C_1, we need to include in V', besides v_1^1, at least two other vertices of C_1. So, two of these vertices are neighbours in C_1, assume w.l.g. v_1^3, v_1^4. But then, vertex v_1^5 should also be contained in V', for otherwise edge (v_1^4, v_1^5) contained in E' has $\mathsf{VP}_{\mathbf{s}}(v_1^4, v_1^5)) = \mathsf{VP}_{\mathbf{s}}(v_1^4) < \mathsf{VP}_{\mathbf{s}}(v_1^3, v_1^4))$. Applying same arguments and for v_1^2, we get that the vertex should also be contained in V'. Summing up, we get that $V(C_1) \subseteq V'$, as required for condition (iv). Moreover, since for each such vertex there exists a neighbour to it vertex also in V', we get that $V(C_1) \subseteq V_r'$. This combined with that $C_1 \subseteq E'$ (Claim E.1), concludes that $C_1 \subseteq G(E'((V_r'))$, as required for condition (v). Finally, note that since $r = 2$ we also get that actually $\mathsf{Neigh}_{G(E')}(v_1^1) = \{v_1^2, v_1^5\}$, as required for condition (vi).

E.2 Proof of Lemma 7.4

Note first that any $v \in V(G^1)$ can not be contained in V_i'. This is so, because v is connected to another vertex contained in V', vertex v_1^1, by Claim E.2. Since also, $V_{C_1}, V_{C_2} \notin V_i'$, by Lemma 7.3, we conclude that actually the subgraph $G(E'(V_{out} \cup V_i'))$ of the characterization of a Defender Uniform Nash equilibrium, Theorem 7.1 is an empty graph. Recalling that E' is an Edge Cover of G, the vertices of G^1 must be covered by an r-regular subgraph of G^1, where $r = 2$, by Lemma 7.3. Finally, since there are no edges between any vertex of the r-regular graph and any vertex of V_{out}, we get that actually all vertices $V(G^1) \in V_r'$.

■

E.3 Proof of Lemma 7.5

Consider a sequence of vertex disjoint directed graphs c, contain all vertices of G_d, i.e., $c = c_1, \cdots, c_k$, where c_j is the j-th cycle of the sequence. We compute a sequence of cycles in G^1 as follows:

1. Consider each directed cycle $c_j \in c$ of G_d, w.l.g. let $c_j = \{(v,u),(u,w),\cdots,(x,z),(z,v)\}$. Compute an undirected cycle c'_j in G^1 as described below. Consider consecutive edges of the cycle c_j, starting from (v,u) and do the following:
   (a) If edge (v,u) is bidirectional, then we add the following path in c'_j, $p_{vu} = \{(v^1,v^2),(v^2,v^3),$ $(v^3,vu^4),$ $(vu^4,e'_{vu}),(e'_{vu},e_{vu}),(e_{vu},e_{uv}),(e_{uv},e'_{uv}),$ $(e'_{uv},u_4),$ $(u_4,u_1),(u_1,u_2),$ $(u_2,u_3)\}$ in c'_j; see Figure 3(ii) for an illustration.
   (b) Otherwise, if edge is unidirectional, add edges $(v^1,v^2),(v^2,v^3),(v^3,u^1),(u_1,u_2),(u_2,u_3)$.
2. Now consider each pair of edges $(u,v),(v,u) \in A_d$. If none of the two edges is contained in c' by now, then add in c' the cycle given by $c_{uv} = \{(vu^4,e'_{vu}),$ $(e'_{vu},e_{vu}),$ $(e_{vu},u_4), uv^44, e'_{uv}),$ $(e'_{uv},e_{uv}),$ $(e_{uv},vu^4)(e_{uv},vu^4)\}$; see Figure 3(iii) for an illustration.

We show that c' is a spanning subgraph of G^1. Note first that for each vertex $v \in V_d$ contained in c, vertices v^1,v^2,v^3 are contained in c' constructed, by rule (1) above. Since c contains all vertices of G_d, all vertices of the form v^1,v^2,v^3, for all $v \in V_d$ are contained in c'. Consider now any pair of edges $(u,v),(v,u) \in A_d$. Note that not both of these edges can be contained in the sequence of cycles c. This is so because the cycles of the sequence are vertex disjoint graphs of size at least 3 each. Thus, there are two cases to consider: either (a) one of the two edges is contained in c, assume w.l.g. (v,u), or (b) none of the two edges in c.
In case (a), the path p_{vu} is contained in the sequence c', by rule 1(a) of the construction of c'. In case (b), by rule 2 of the construction of c', the cycle c_{uv} is contained in c'. In both cases, subgraphs p_{vu} and c_{uv} contains all vu-vertices of G^1. We conclude that for all uv-vertices, for all pairs of edges $(u,v),(v,u) \in A_d$, are contained in c' constructed. This combined with the above observations conclude that all vertices of G^1 are contained in c' constructed.

E.4 Proof of Lemma 7.6

Let c' the sequence of vertex cycles containing all vertices of G^1, i.e. $c' = c'_1,\cdots,c'_{k'}$. In order a vertex of the type $v^2 \in V(G^1)$, for some $v \in V_d$, to be covered by a cycle in c', edges $(v^1,v^2),(v^2,v^3)$ should be contained in c'. Consider such a cycle $c'_j \in c'$ containing these edges. Look the next edge of the cycle. There are two possible cases for it; it is either of type (1) (v^3,vu^4), when edges $(v,u),(u,v) \in A_d$ or of type (2) (v^3,u^1).

1. We show that the next vertex of the cycle must be vertex e'_{vu}: Assuming otherwise, there are two possible cases; the next edge of the cycle is either (I) (vu^4,v^1) or (II) (vu^4,e'_{vu}).
   For case (I), it can be easily seen, that there is no way for vertices e'_{vu} and e'_{uv} to be covered by a vertex disjoint cycle in c', a contradiction.
   For case (II), by the construction of G^1, the next edges of the cycle are, in order the following: $(e_{vu},e_{uv}),(e_{uv},e'_{uv}),$ $(e'_{uv},u_4),$ $(u_4,u_1),(u_1,u_2),$ (u_2,u_3), since any other case would exclude a vu-vertex from being covered by c'.
   Thus, in this case, cycle c_j contains a p_{vu} path, and it can be illustrated as in Figure 2(c)(ii).
2. When (v^3,u^1) is the next edge of c_j, then the following edges of the cycle are avoidably edges $(u^3,u^2),(u^2,u^3)$, otherwise vertex u^2 could not be covered by any other cycle of c'.

Continue with the next edge of the cycle. Again there are two possible cases for it; it is either of type (1) or (2) as described above.

Now, from c', we construct a set of directed edges c in G^d as follows. For each $v \in V_d$, consider any edge of c' containing vertex $v^3 \in V(G^1)$. By the above remarks, the edge can be either of type (v^3, u^1), or of type (v^3, vu^4). Then, we add edge (v, u) in c. Note that, G_d contains such an edge, by its construction of G^1. Thus, $c \subseteq A$. Moreover, by the structure of any cycle of c', shown in cases 1 and 2 above, we conclude that the set of edges c' constitute a set of vertex disjoint cycles of G_d. Finally, since each vertex of type v^1, for any $v \in V_d$, is contained in c', c constructed contains all vertices of $v \in V_d$.

E.5 Proof of Lemma 7.7

Consider a set of vertex disjoint directed cycles contained in G_d, c, containing all of its vertices. Then, by Lemma 7.5, G^1 contains vertex disjoint cycles containing all of its vertices, assume any such sequence c'. Compute the following profile of $\Pi(G)$, $\mathbf{s}$: Set $\mathsf{Support}_{\mathbf{s}}(ep) = C_1 \cup C_2 \cup c'$ and $\mathsf{Support}_{\mathbf{s}}(vp) = V(C_1) \cup V(c) \cup V(C_2)$. Set, $P_{\mathbf{s}}(ep, e) = \frac{1}{\mathsf{Support}_{\mathbf{s}}(ep)}$, $\forall e, \in E$ and $s_i(v) = \frac{1}{\mathsf{Support}_{\mathbf{s}}(vp)}$, $\forall v \in V$ and all $vp_i \in \mathcal{N}_{vp}$. Note that $\mathbf{s}$ is a Defender Uniform Nash Equilibrium profile. Also that, $\Delta_{G(E')} = 2$, as required by Lemma 7.3. It can be easily seen than all conditions of Theorem 2.1 are satisfied, thus it is a NE.

E.6 Proof of Lemma 7.8

By Lemma 7.3, $r = 2$. Also, by the same Lemma, $C_2 \subseteq G(E')$ and $V(C_2) \subseteq V'_r$ and $V(C_1) \subseteq V'_r$, $C_1 \subseteq G((E'(V'_r))$ and that by Lemma 7.4, $V(G^1) \in V'_r$. Thus, G^1 contains a r-regular graph containing all of its vertices, which is contained in $\mathsf{Support}_{\mathbf{s}}(ep)$. That is G^1 contains a sequence of vertex disjoint cycles, c', containing all of its vertices. Then by Lemma 7.6, the DIRECTED PARTITION INTO HAMILTONIAN SUBGRAPHS problem in G_d is answered positively.

F Proof of Theorem 8.1

We first show that if at least one of cases (1) or (2) hold then $\Pi(G)$ contains an attacker symmetric uniform NE. We show the claim when each one of the two cases hold:

Case 1. Construct an attacker symmetric uniform profile $\mathbf{s}$ such that $\mathsf{Support}_{\mathbf{s}}(vp) := V$ and $P_{\mathbf{s}}(e) := p(e)$, for all $e \in E$. Since p is a probability distribution, $\mathbf{s}$ is a valid mixed profile of $\Pi(G)$. We prove that all conditions of Theorem 2.1 hold for $\mathbf{s}$, i.e. it is a NE. Condition 1/b implies that any vertex $v \in V$ is hit a with positive probability equal to $P_{\mathbf{s}}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges}_G(v)} P_{\mathbf{s}}(e) = \sum_{e \in \mathsf{Edges}_G(v)} p(e)$. Condition 1/b guarantees that $P_{\mathbf{s}}(\mathsf{Hit}(v)) = P_{\mathbf{s}}(\mathsf{Hit}(v'))$, for any two vertices v and $v' \in V$, which proves Condition (1) of Theorem 2.1.

By Condition 1/b, it follows $\mathsf{Support}_{\mathbf{s}}(vp) = V$. Alos, since $\mathbf{s}$ is an attacker symmetric uniform profile, for any $e = (u, v) \in E$, $\mathsf{VP}_{\mathbf{s}}(e) = \mathsf{VP}_{\mathbf{s}}(v) + \mathsf{VP}_{\mathbf{s}}(u) = \sum_{vp_i \in \mathcal{N}_{vp}} \frac{1}{|\mathsf{Support}_{\mathbf{s}}(vp)|} + \sum_{vp_i \in \mathcal{N}_{vp}} \frac{1}{|\mathsf{Support}_{\mathbf{s}}(vp)|} = \frac{2\nu}{|V|}$. It follows that, $\mathsf{VP}_{\mathbf{s}}(e) = \max_{e \in E} \mathsf{VP}_{\mathbf{s}}(e)$, for any $e \in \mathsf{Support}_{\mathbf{s}}(ep)$, proving Condition (2) of Theorem 2.1.

Case 2. Theorem 2.2 implies that G contains a Matching NE, which is, by its definition, an attacker symmetric uniform NE.

It follows that if at least one of cases (1) or (2) hold then $\Pi(G)$ contains an attacker symmetric uniform NE.

We proceed to show that if G admits an Attacker Symmetric Uniform NE then at least one of cases (1) or (2) of the Theorem hold. Consider an attacker symmetric uniform NE $\mathbf{s}$. There are two cases for set $\mathsf{Support}_{\mathbf{s}}(vp)$: either (i) there exist two vertices v and u of $\mathsf{Support}_{\mathbf{s}}(vp)$ such that $e = (v, u) \in E$

or (ii) for any two vertices v and u of $\mathsf{Support_s}(vp)$, it holds that $e = (v, u) \notin E$. We show that when case (i) holds, Case (1) holds and when (ii) holds, Case (2) holds.

Case (i): Observe that for any edge $(v, u) = e \in E$, for which $v, u \in \mathsf{Support_s}(vp)$, it holds that $\mathsf{VP_s}(e) = \mathsf{VP_s}(v) + \mathsf{VP_s}(u) = \sum_{vp_i \in \mathcal{N}_{vp}} \frac{1}{|\mathsf{Support_s}(vp)|} + \sum_{vp_i \in \mathcal{N}_{vp}} \frac{1}{|\mathsf{Support_s}(vp)|} = \frac{2\nu}{|\mathsf{Support_s}(vp)|}$, by the definition of an attacker symmetric uniform profile. We first show that $\mathsf{Support_s}(vp) = V$. Assume, in contrary, that there exists a vertex $v' \notin \mathsf{Support_s}(vp)$. Then consider an edge $e' = (u', v') \in \mathsf{Support_s}(ep)$, for some $u' \in V$; such an edge exists by the Covering Conditions of any NE. For edge e' it holds that $\mathsf{VP_s}(e') = \mathsf{VP_s}(v') + \mathsf{VP_s}(u') = \frac{\nu}{|\mathsf{Support_s}(vp)|} < \mathsf{VP_s}(e)$. A contradiction to that Condition (2) of the characterization of a NE regarding edges e and e'. It follows that $\mathsf{Support_s}(vp) = V$.

We next show that setting $p(v) := P_{\mathsf{s}}(v)$, for all $v \in V$, we get a feasible solution of the Linear System $p(\cdot)$. Condition (1) of the characterization of a NE implies that for any $v \in \mathsf{Support_s}(vp)$, $P_{\mathsf{s}}(\mathsf{Hit}(v)) = \sum_{e \in \mathsf{Edges}_G(v)} P_{\mathsf{s}}(e) = \min_{v' \in V} P_{\mathsf{s}}(\mathsf{Hit}(v'))$. Thus, $\sum_{e \in \mathsf{Edges}_G(v)} p(e) = \min_{v' \in V} \sum_{e' \in \mathsf{Edges}_G(v')} p(e')$, for any $v \in \mathsf{Support_s}(vp)$. Since $V = \mathsf{Support_s}(vp)$, Condition 1/a of function $p(\cdot)$ is satisfied. Moreover, since $\mathsf{Support_s}(ep)$ is an Edge Cover of G, for any $v \in V$, $\sum_{e \in \mathsf{Edges}_G(v)} P_{\mathsf{s}}(e) > 0$ and hence Condition 1/b of function $p(\cdot)$ is also satisfied. It follows that the assignment of $p(\cdot)$ is a feasible solution of the linear system $p(\cdot)$ and so Case (1) of the Theorem holds.

Case (ii): Let $IS := \mathsf{Support_s}(vp)$. IS is an Independent Set of G, by assumption. We show that IS is actually an Expanding Independent Set of G. Assume, the contrary. Then, there exists a set $A_1 \subseteq V \backslash IS$ such that $|\mathsf{Neigh}_G(A_1) \cap IS| < |A_1|$. Note that is also $\mathsf{Support_s}(vp)$ is a Vertex Cover of $G(\mathsf{Support_s}(ep))$. It follows that for each edge $e \in \mathsf{Support_s}(ep)$, exactly one of its endpoints is contained in $\mathsf{Support_s}(vp)$.

Thus, regarding vertex set $\mathsf{Neigh}_G(A_1) \cap IS \subseteq IS$, it holds that

$$\sum_{v \in \mathsf{Neigh}_G(A_1) \cap IS} P_{\mathsf{s}}(\mathsf{Hit}(v)) = \sum_{v \in \mathsf{Neigh}_G(A_1) \cap IS} \sum_{e \in \mathsf{Edges_s}(v)} s_{ep}(e) = \sum_{v \in \mathsf{Neigh}_G(A_1) \cap IS} s_{ep}(e) = P_1.$$

Since for each $v \in \mathsf{Support_s}(v)$, $P_{\mathsf{s}}(\mathsf{Hit}(v)) = \min_{v \in V} P_{\mathsf{s}}(\mathsf{Hit}(v))$, it follows that $P_1 \leq |\mathsf{Neigh}_G(A_1) \cap IS| \cdot \min_{v \in V} P_{\mathsf{s}}(\mathsf{Hit}(v))$.

Also, regarding the vertex set $A_1 \subseteq V \backslash IS$, it holds that

$$\sum_{v \in A_1} P_{\mathsf{s}}(\mathsf{Hit}(v)) = \sum_{v \in A_1} \sum_{e \in \mathsf{Edges_s}(v)} s_{ep}(e) = \sum_{v \in A_1} P(\mathsf{Hit}(v)) s_{ep}(e) = P_1.$$

We argue that in any probability assignment on $\mathsf{Edges_s}(A_1)$, there exists a vertex $v' \in A_1$, such that $P_{\mathsf{s}}(\mathsf{Hit}(v')) < \min_{v \in V} P_{\mathsf{s}}(\mathsf{Hit}(v))$, which contradicts to Condition 1 of the characterization of a NE. The probabilities assignment problem on $\mathsf{Edges_s}(A_1)$ is equivalent to the problem of distributing a quantity of $P_1 \leq |\mathsf{Neigh}_G(A_1) \cap IS| \cdot \min_{v \in V} P_{\mathsf{s}}(\mathsf{Hit}(v))$ to a set of $|A_1|$ distinct bins. Since $|\mathsf{Neigh}_G(A_1) \cap IS| < |A_1|$, by assumption, it follows that in any assignment there must be a vertex $u \in A_1$ such that $\sum_{e \in \mathsf{Edges_s}(u)} s_{ep}(e) = P_{\mathsf{s}}(\mathsf{Hit}(v)) < \min_{v \in V} P_{\mathsf{s}}(\mathsf{Hit}(v))$, as claimed.

It follows that if at least one of cases (1) or (2) hold then G contains an attacker symmetric uniform NE, which concludes the Theorem.